Image Transforms summary

Prof. Ioannis Pitas
Aristotle University of Thessaloniki
pitas@csd.auth.gr
www.aiia.csd.auth.gr
Version 3.1

Artificial Intelligence & Information Analysis Lab

Contents

2D Discrete Space Fourier Transform

2D discrete space Fourier transform is defined as:

$$X(\omega_1,\omega_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x(n_1,n_2)\exp(-i\omega_1 n_1 - i\omega_2 n_2)$$

$$x(n_1,n_2) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\omega_1,\omega_2)\exp(i\omega_1 n_1 + i\omega_2 n_2)\,d\omega_1 d\omega_2 \,.$$

Introduction

Image transforms are represented by transform matrices $\mathbf{A}$:

$$\mathbf{X} = \mathbf{A}\mathbf{x},$$

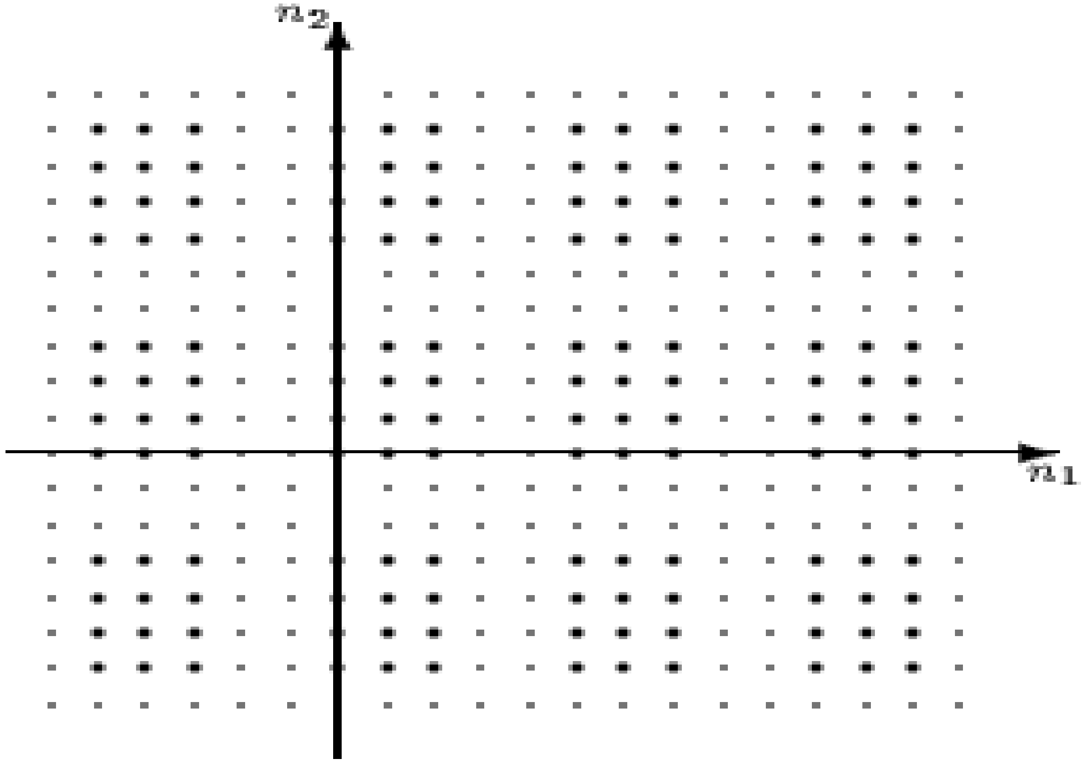
- $\mathbf{x}$ and $\mathbf{X}$ are the original and transformed image respectively.

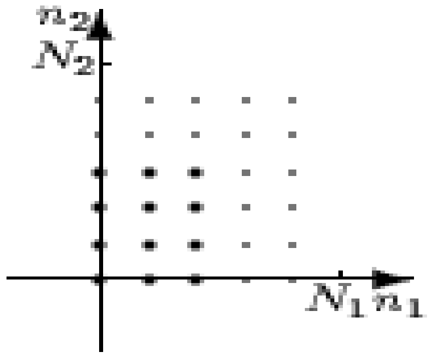
In most cases the transform matrices are ***unitary***:

$$\mathbf{A}^{-1} = \mathbf{A}^{*T}.$$

- The columns of $\mathbf{A}^{*T}$ are the ***basis vectors*** of the transform.
- In image transforms, they correspond to ***basis images***.
- The most popular image transforms are:
  - Discrete Fourier Transform (DFT).
  - Discrete Cosine Transform (DCT).

2D Discrete Fourier Transform





Rectangularly periodic sequence and its fundamental period.

Artificial Intelligence &
Information Analysis Lab

2D Discrete Fourier Transform

- A discrete 2D space-limited signal can be represented by DFT as:

$$X(k_1,k_2)=\sum_{n_1=0}^{N_1-1}\sum_{n_2=0}^{N_2-1}x(n_1,\mathrm{n}_2)\exp(-i\frac{2\pi n_1k_1}{N_1}-i\frac{2\pi n_2k_2}{N_2})$$

$$x(n_1,n_2)=\frac{1}{N_1N_2}\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1}X(k_1,k_2)\exp(i\frac{2\pi n_1k_1}{N_1}+i\frac{2\pi n_2k_2}{N_2}).$$

Artificial Intelligence & Information Analysis Lab

2D Discrete Fourier Transform

2D cyclic signal shift is defined by:

$$y(n_1,n_2) = x\left(((n_1 - m_1))_{N_1}, ((n_2 - m_2))_{N_2}\right),$$

$$((n))_N \triangleq n \bmod N.$$

2D cyclic convolution of two signals can be computed by means of the cyclic shift of one of the two signals:

$$y(n_1,n_2) \triangleq x(n_1,n_2) \circledast\circledast h(n_1,n_2) =$$

$$= \sum_{m_1=0}^{N_1} \sum_{m_2}^{N_2-1} x(m_1,m_2) h\left(((n_1 - m_1))_{N_1}, ((n_2 - m_2))_{N_2}\right).$$

2D Discrete Fourier Transform

- ***Cyclic convolution***:

$$x(n_1, n_2) \circledast\circledast h(n_1, n_2) \leftrightarrow X(k_1, k_2) \cdot H(k_1, k_2),$$

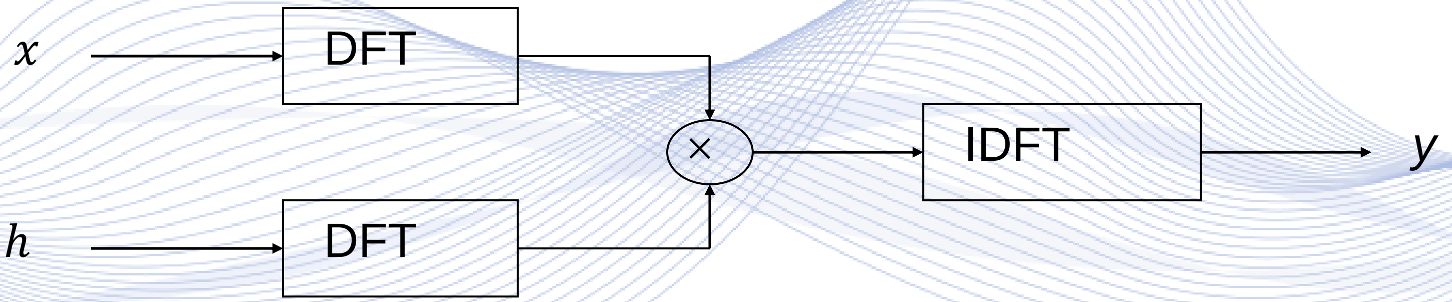
- Signal multiplication:

$$x(n_1, n_2)h(n_1, n_2) \leftrightarrow \frac{1}{N_1 N_2} X(k_1, k_2) \circledast\circledast H(k_1, k_2).$$

2D Discrete Fourier Transform

3. Calculate the DFTs of the new sequences $x_p(n_1, n_2)$ and $h_p(n_1, n_2)$.
4. Calculate the DFT $Y_p(k_1, k_2)$, as the product of $X_p(k_1, k_2)$ and $H_p(k_1, k_2)$.
5. Calculate $y_p(n_1, n_2)$ by using the inverse DFT. The result of the linear convolution is:

$$y(n_1, n_2) = y_p(n_1, n_2), \qquad (n_1, n_2) \in \mathcal{R}_{L_1 L_2}$$



Convolution calculation using DFTs.

Row-Column FFT algorithm

- The simplest algorithm for the calculation of the 2D DFT is the ***Row-Column FFT*** (***RCFFT***) algorithm.

- The 2D DFT can be decomposed in N_1 DFTs along rows and N_2 DFTs along columns:

$$X'(n_1, k_2) = \sum_{n_2=0}^{N_2-1} X(n_1, n_2)\, W_{N_2}^{n_2 k_2}$$

$$X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} X'(n_1, k_2)\, W_{N_1}^{n_1 k_1}.$$

Row-Column FFT algorithm

- The number of complex multiplications for RCFFT is:

$$C = N_1 \frac{N_2}{2} \log_2 N_2 + N_2 \frac{N_1}{2} \log_2 N_1 = \frac{N_1 N_2}{2} \log_2 (N_1 N_2).$$

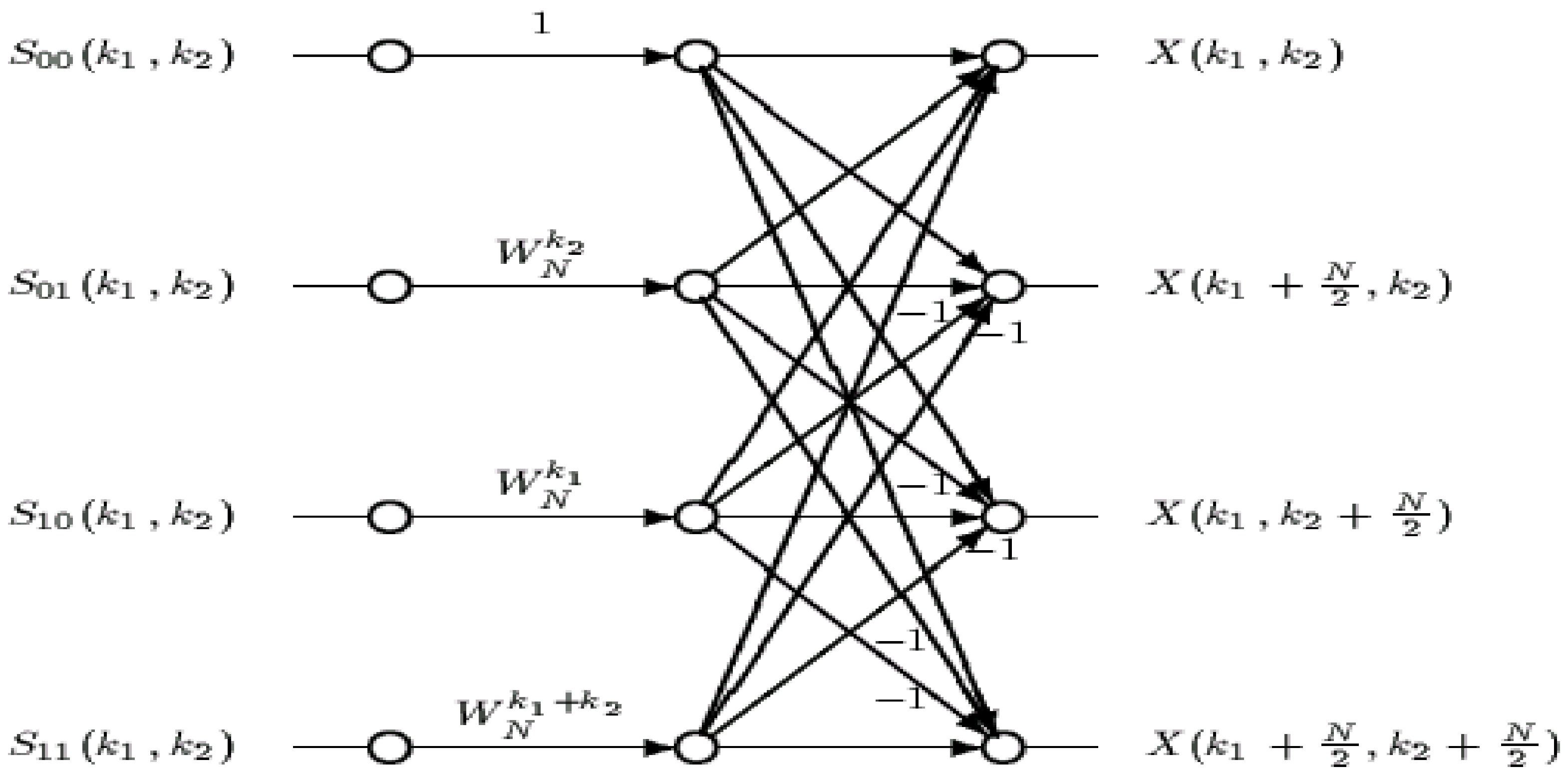
- If radix-2 FFT is used then the number of complex additions for RCFFT is:

$$A = N_1 N_2 \log_2 (N_1 N_2).$$

- The computational complexity is of the order :

$$O(kN^2 \log_2 N)$$

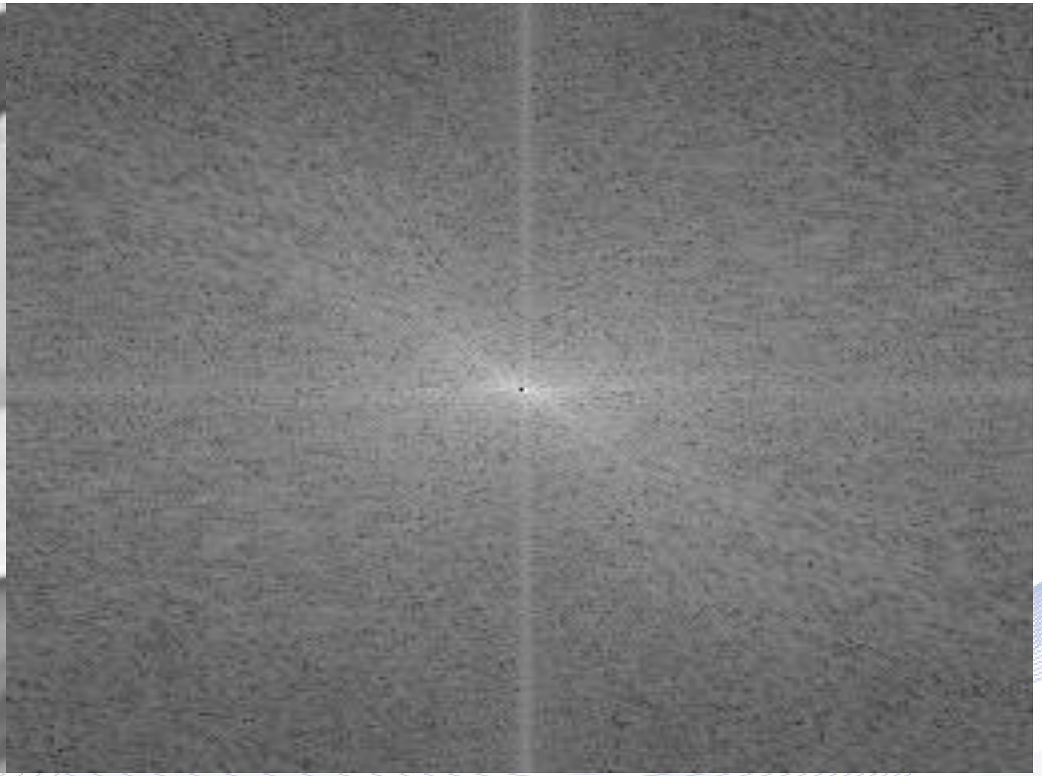
Vector-radix FFT algorithm



Radix 2×2 butterfly.

2D Power Spectrum estimation


a) Image LENNA;


b) periodogram of LENNA.

Discrete Cosine Transform

- DCT is used in the JPEG and MPEG standards.
- Forward DCT transform:

$$C(0) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n)$$

$$C(k) = \sqrt{\frac{2}{N}} \sum_{n=0}^{N-1} x(n) \cos \frac{(2n+1)k\pi}{2N}.$$

- Inverse DCT:

$$x(n) = \frac{1}{\sqrt{N}} C(0) + \sqrt{\frac{2}{N}} \sum_{k=1}^{N-1} C(k) \cos \frac{(2n+1)k\pi}{2N}.$$

Q & A

Thank you very much for your attention!

Contact: Prof. I. Pitas
pitas@csd.auth.gr