Two-Dimensional Systems

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Artificial Intelligence &
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- Two-Dimensional Discrete Systems.
- Two-Dimensional $\mathcal{Z}$ Transform.
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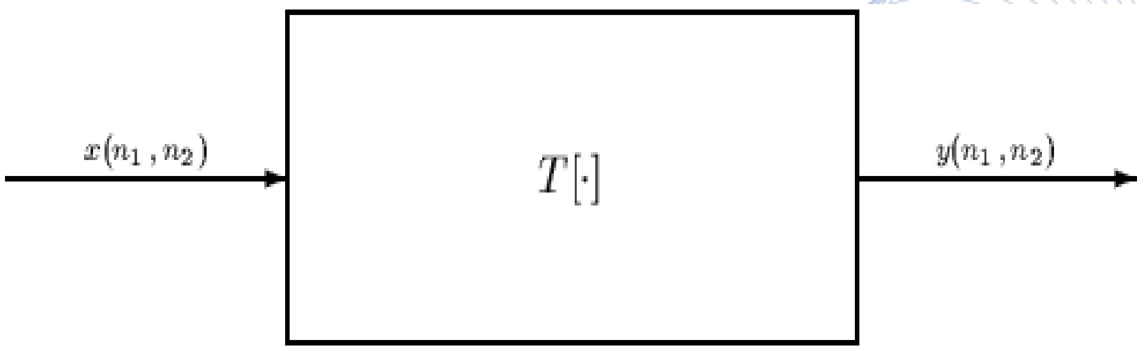
2D Discrete Systems

- Definition: A Two-Dimensional (2D) Discrete System T transforms a 2D signal $x(n_1, n_2)$ to a 2D output signal $y(n_1, n_2)$:

$$y(n_1, n_2) = T[x(n_1, n_2)].$$



2D Discrete System.

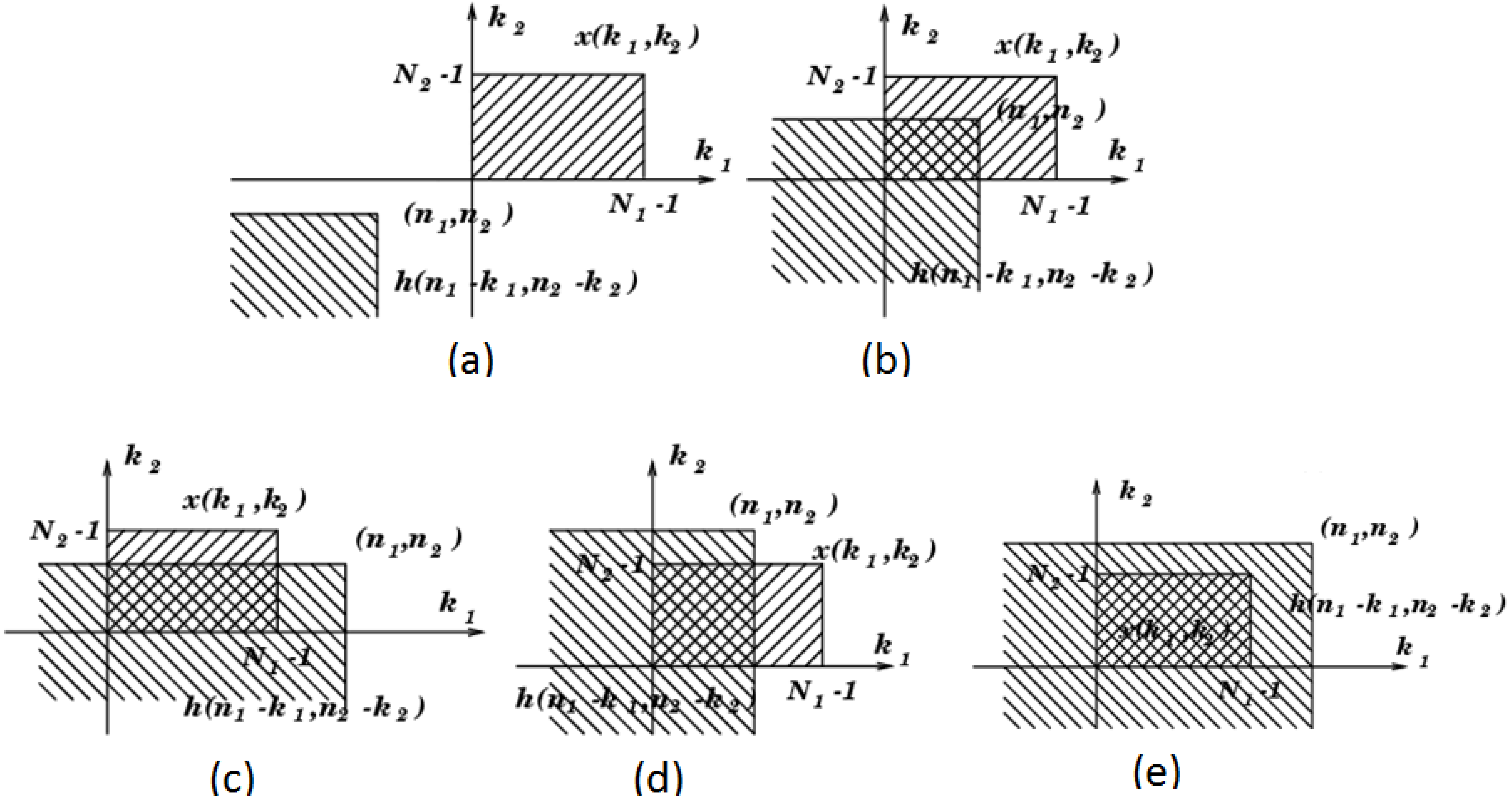
2D Discrete Systems

A 2D ***Linear Spatially Invariant*** (***LSI***) system is described by a ***2D linear convolution***:

$$y(n_1,n_2) = x(n_1,n_2) ** h(n_1,n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1,k_2)h(n_1-k_1,n_2-k_2).$$

An LSI system is described by its ***2D impulse response*** $h(n_1,n_2)$.

2D Discrete Systems



Visualization of 2D convolution calculation.

2D linear correlation

2D ***correlation*** of template image h and input image x (inner product):

$$r_{hx}(n_1,n_2) = \sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} h(k_1,k_2)\,x(k_1+n_1,k_2+n_2) = \mathbf{h}^T\mathbf{x}(n_1,n_2).$$

- $\mathbf{h} = [h(0,0),\dots,h(N_1-1,N_2-1)]^T$: template image vector.
- $\mathbf{x}(n_1,n_2) = [x(n_1,n_2),\dots,x(n_1+N_1-1,n_2+N_2-1)]^T$: local neighborhood (window) image vector.

2D linear correlation

Differences from convolution:

- $x(n_1,n_2)$ is not flipped around $(0,0)$.
- ***It is often confused with convolution***: they are identical only if h is centered at and is symmetric about $(0,0)$.
- It is used for 2D template matching and for object detection and tracking in video.

Image autocorrelation:

$$r_{xx}(n_1,n_2)=\sum_{k_1=0}^{N_1-1}\sum_{k_2=0}^{N_2-1} x(k_1,k_2)\,x(k_1+n_1,k_2+n_2).$$

2D Discrete Systems

The impulse response support distinguishes 2D systems in:

- ***2D Finite Impulse Response*** (***FIR***) systems have a finite filter window of size $M_1 \times M_2$ samples: $0 \leq n_1 < M_1, \quad 0 \leq n_2 < M_2$.

They are described by the 2D convolution:

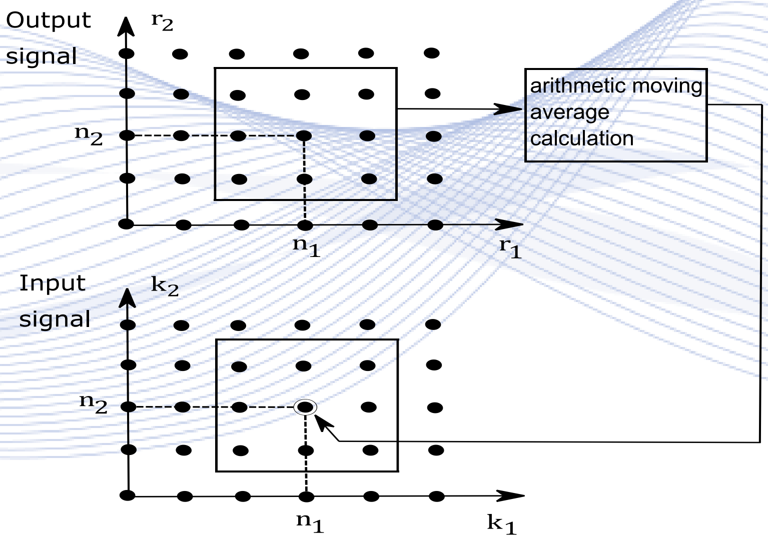
$$y(n_1, n_2) = \sum_{k_1=0}^{M_1-1} \sum_{k_2=0}^{M_2-1} h(k_1, k_2) x(n_1 - k_1, n_2 - k_2).$$

2D Discrete Systems

Example of an FIR is the arithmetic moving average filter:

$$y(n_1, n_2) = \frac{1}{M_1 M_2} \sum_{k_1=-v_1}^{v_1} \sum_{k_2=-v_2}^{v_2} x(n_1 - k_1, n_2 - k_2).$$

- Odd window size: $M_i = 2v_i + 1, \quad i = 1,2.$



2D Discrete Systems

Moving average image filtering.

2D Discrete Systems

2D IIR filter (edge detector) output.

2D $\mathcal{Z}$ Transform

- Definition of ***2D $\mathcal{Z}$ Transform***:

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) z_1^{-n_1} z_2^{-n_2}.$$

- It performs a mapping from $\mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$.
- It can be considered as a 2-variable polynomial of $z_1^{-1} z_2^{-1}$.
- Definition of reverse 2D $\mathcal{Z}$ Transform:

$$x(n_1, n_2) = \left(\frac{1}{2\pi i}\right)^2 \oint_{C_1} \oint_{C_2} X(z_1, z_2) z_1^{n_1-1} z_2^{n_2-1} dz_1 dz_2.$$

2D $\mathcal{Z}$ Transform

Reflection about an axis:

$$x(-n_1, n_2) \leftrightarrow X(z_1^{-1}, z_2),$$
$$x(n_1, -n_2) \leftrightarrow X(z_1, z_2^{-1}),$$
$$x(-n_1, -n_2) \leftrightarrow X(z_1^{-1}, z_2^{-1}).$$

Convolution:

$$y(n_1, n_2) = x(n_1, n_2) ** h(n_1, n_2) \leftrightarrow Y(z_1, z_2) = X(z_1, z_2)H(z_1, z_2).$$

Transfer Function of 2D Digital Filters

- ***2D transfer function*** definition:

$$H(z_1, z_2) = \frac{Y(z_1, z_2)}{X(z_1, z_2)}.$$

- Transfer function of a 2D FIR filter:

$$H(z_1, z_2) = \sum_{k_1=0}^{M_1-1} \sum_{k_2=0}^{M_2-1} h(k_1, k_2) z_1^{-k_1} z_2^{-k_2}.$$

- It is a 2 variable polynomial of z_1^{-1}, z_2^{-1}.
- Such polynomials: a) can not be easily factorized; b) they do not have distinct roots.

Transfer Function of 2D Digital Filters

- Transfer function of a 2D IIR filter:

$$H(z_1, z_2) = \frac{\sum_{r_1}\sum_{r_2} a(r_1, r_2) z_1^{-r_1} z_2^{-r_2}}{\sum_{k_1}\sum_{k_2} b(k_1, k_2) z_1^{-k_1} z_2^{-k_2}} = \frac{A(z_1, z_2)}{B(z_1, z_2)}.$$

- It is a 2D rational function of z_1^{-1}, z_2^{-1}.
- Denominator polynomial $B(z_1, z_2)$ may become 0, leading the IIR system to instability. It cannot be easily factorized.

Implementation of 2D Digital Filters

- 2D IIR filters employ both input and past output pixels for computing current output pixel.
- This poses constraints on the IIR output filter mask.
- 2D IIR filters split the image plane in ***past***, ***current*** and ***future*** pixels to be visited.
- Their definition depends on the way image is scanned.
- Maximal output filter mask: half-plane one.

Q & A

Thank you very much for your attention!

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