Fuzzy Scalar and Vector Median filters based on Fuzzy Distances

Vassilios Chatzis and Ioannis Pitas*

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Abstract

In this paper, the Fuzzy Scalar Median (FSM) is proposed, defined by using ordering of fuzzy numbers based on fuzzy minimum and maximum operations defined by using the extension principle. Alternatively, the FSM is defined from the minimization of a fuzzy distance measure, and the equivalence of the two definitions is proven. Then, the Fuzzy Vector Median (FVM) is proposed as an extension of Vector Median, based on a novel distance definition of fuzzy vectors, which satisfy the property of angle decomposition. By defining properly the fuzziness of a value, the combination of the basic properties of the classical Scalar and Vector Median (VM) filter with other desirable characteristics can be succeeded.

1 Introduction

The median operator have been extensively used as a filter in digital signal and image processing [1]. It combines the desirable characteristics of the impulsive noise removal and the preservation of the signal or image edges. Since multichannel signals appear in many important signal processing applications, multichannel techniques, that have been proposed rather recently and take into account channel correlation, seem to be the most appropriate way to process such signals. One of the most popular technique to process a multichannel signal corrupted by impulsive noise is the Vector Median (VM) filter, that inherently utilizes the correlation of the channels and keeps the desirable properties of the scalar median, the zero impulse response and the preservation of the signal edges [5].

In this paper the Fuzzy Scalar Median (FSM) and the Fuzzy Vector Median (FVM) will be proposed, in order to utilize the fuzziness of the provided information. Fuzziness can be generally considered as a measure of the unreliability of a value. It can be either provided to accompany a classical value, or estimated by the classical values. Several applications can be found in the domain of image

*The authors are with the Department of Informatics, Aristotle University of Thessaloniki. Corresponding author: Ioannis Pitas, University of Thessaloniki, GR-54006 Thessaloniki, GREECE, phone,fax: +30-31-996304, e-mails: [chatzis,pitas]@zeus.csd.auth.gr

processing. Fuzziness can describe ambiguity regarding a pixels chromatic value or location, that can be due to the recording devices, transmission errors, lighting conditions etc. Fuzziness can be either provided by the devices, or estimated by expert systems using the chromatic values of a pixel or a pixels neighborhood. The concealed uncertainty, which is supposed in any scalar or vector value, will be taken into account by using fuzzy numbers or fuzzy vectors respectively.

The Fuzzy Scalar Median is proposed, first as an extension of classical median operator, and then defined by minimizing a fuzzy distance function. The equivalence of the definitions is proven. Then, the angle decomposed fuzzy vectors are defined as multidimensional fuzzy sets with special properties, and a distance between them is proposed. The Fuzzy Vector Median is defined and its basic properties are presented. A proper definition of the fuzziness of a value, can lead to a desirable combination of the classical Vector Median (VM) properties with other features.

2 The Fuzzy Scalar Median (FSM) defined through fuzzy ordering

The α-cuts of a fuzzy set are the classical sets X^α, where $x \in X^\alpha \Leftrightarrow \mu(x) \geq \alpha$, $\alpha \in [0,1]$. A fuzzy set is called *normal* if $\exists \mathbf{x} : \mu(\mathbf{x}) = 1$ or $\mathbf{X}^1 \neq \emptyset$. It is called *convex* if $\forall \alpha_1, \alpha_2 \in [0,1], \alpha_1 > \alpha_2 \Leftrightarrow \mathbf{X}^{\alpha_1} \subseteq \mathbf{X}^{\alpha_2}$. A normal and convex fuzzy set is called *fuzzy number* [6]-[8]. A 1-dimensional fuzzy number will be called *convex fuzzy number* when the corresponding α-cuts are convex sets.

The maximum MAX and minimum MIN of two fuzzy numbers X_1, X_2, fuzzified by using the extension principle [2, 7], and presented by their α-cuts, are given by [7]:

$$\mathrm{MAX}\{X_1, X_2\} = \bigcup_\alpha [\max\{x_{1_l}^\alpha, x_{2_l}^\alpha\}, \max\{x_{1_r}^\alpha, x_{2_r}^\alpha\}] \tag{1}$$

$$\mathrm{MIN}\{X_1, X_2\} = \bigcup_\alpha [\min\{x_{1_l}^\alpha, x_{2_l}^\alpha\}, \min\{x_{1_r}^\alpha, x_{2_r}^\alpha\}] \tag{2}$$

An example is shown in Figure 1 where the fuzzy minimum and fuzzy maximum of two fuzzy numbers X_1, X_2 shown in Figure 1a, are shown in Figure 1b. This example emphasizes that fuzzy maximum or minimum of two fuzzy numbers is not always one of the two numbers, as in crisp arithmetic. Any order statistic of n fuzzy numbers $X_1, X_2, \ldots, X_n$ can be found by using successive fuzzy maximum and minimum operators. It is easy to prove, by using equations (1),(2), that successive MAX and MIN operations can be calculated by performing the corresponding crisp successive *max*, *min* operations on the crisp limits of the α-cuts of the fuzzy numbers. The ordered fuzzy numbers of n fuzzy numbers can be calculated as the union of the corresponding ordered α-cuts. The ordered $\alpha - cuts$ are calculated using crisp *max*, *min* operations on the limits of the α-cuts of the fuzzy numbers. Let the ordered fuzzy numbers be symbolized as $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where $X_{(1)}$ is the fuzzy minimum and $X_{(n)}$ the fuzzy maximum. The ordered fuzzy numbers are also written as:

$$X_{(i)} = \bigcup_{\alpha} [x^{\alpha}_{(i)_l}, x^{\alpha}_{(i)_r}], \qquad i = 1, 2, \ldots, n \tag{3}$$

where $x^{\alpha}_{(i)_l}$ are the order statistics of $x^{\alpha}_{i_l}$ and $x^{\alpha}_{(i)_r}$ are the order statistics of $x^{\alpha}_{i_r}$. Similarly with the crisp case we can call the ordered fuzzy numbers $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ as *fuzzy order statistics* of the fuzzy numbers $X_1, X_2, \ldots, X_n$. Then, the median M_o defined from the order statistics of n fuzzy numbers is given by:

$$M_o\{X_1, X_2, \ldots, X_n\} = \bigcup_{\alpha} [x^{\alpha}_{(\frac{n+1}{2})_l}, x^{\alpha}_{(\frac{n+1}{2})_r}] \tag{4}$$

The above equation is valid when n is odd. If n is even the median can be defined as the $\frac{n}{2}$ or the $\frac{n}{2}+1$ order statistic $X_{(\frac{n}{2})}$ or $X_{(\frac{n}{2}+1)}$ respectively, or as their arithmetic mean $\frac{1}{2}(X_{(\frac{n}{2})} + X_{(\frac{n}{2}+1)})$.

Any fuzzy order statistic is not generally one of the input fuzzy numbers as in crisp arithmetic. For example let X_1, X_2, X_3, X_4, X_5 be five fuzzy numbers illustrated in Figure 2a. Fuzzy order statistics $X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}$ are illustrated in Figure 2b. In this example $X_{(3)}$ is the fuzzy median and only the fuzzy maximum $X_{(5)}$ is equal to X_3 of the input fuzzy numbers.

3 The FSM defined through the minimization of a fuzzy distance

An alternative definition of the fuzzy median, based on the minimization of a fuzzy distance measure, will be proposed in the following. It is well known that the median of a set of crisp numbers $x_1, x_2, \ldots, x_n$ minimizes the error function: $\sum_{i=1}^{n} |x_i - m|$, based on the L_1 distance norm. An extension of the L_1 distance norm to fuzzy numbers, is the norm $d_1[.,.]$ that corresponds to the distance of two fuzzy numbers X and Y defined as [6]:

$$d_1[X, Y] = \frac{1}{2} \int_{\alpha=0}^{1} (|x^{\alpha}_l - y^{\alpha}_l| + |x^{\alpha}_r - y^{\alpha}_r|) d\alpha \tag{5}$$

which is the integral of the distances of the corresponding α-cuts. Then, the fuzzy median can be defined by using (5).

Definition 1 *The Fuzzy Scalar Median (FSM) of $X_1, X_2, \ldots, X_n$ fuzzy numbers is the fuzzy number $M = \bigcup_\alpha [m_l^\alpha, m_r^\alpha]$ which minimizes the expression $E = \sum_{i=1}^{n} d_1[X_i, M]$*

By using relation (5) the error function E is now a crisp function, written as:

$$E = \int_{\alpha=0}^{1} \sum_{i=1}^{n} (|x_{i_l}^\alpha - m_l^\alpha| + |x_{i_r}^\alpha - m_r^\alpha|) d\alpha \tag{6}$$

If we denote by E_α the sums: $E_\alpha = \sum_{i=1}^{n} |x_{i_l}^\alpha - m_l^\alpha| + |x_{i_r}^\alpha - m_r^\alpha|$, the minimization of each E_α is equivalent to the minimization of E since E_α are all positive. The parameters m_l^α, m_r^α are determined by crisp minimization as [3]: $m_l^\alpha = \text{median}\{x_{1_l}^\alpha,\ x_{2_l}^\alpha, \ldots\ x_{n_l}^\alpha\}$ and $m_r^\alpha = \text{median}\{x_{1_r}^\alpha,\ x_{2_r}^\alpha, \ldots\ x_{n_r}^\alpha\}$ $\forall \alpha \in [0, 1]$. Thus, the FSM is given by:

$$M = \bigcup_\alpha [\text{median}\{x_{1_l}^\alpha, x_{2_l}^\alpha, \ldots, x_{n_l}^\alpha\}, \text{median}\{x_{1_r}^\alpha, x_{2_r}^\alpha, \ldots, x_{n_r}^\alpha\}] \tag{7}$$

If n is odd, the median of the n crisp lower $x_{i_l}^\alpha$ and upper $x_{i_r}^\alpha$ limits, are the numbers $x_{(\frac{n+1}{2})_l}^\alpha$ and $x_{(\frac{n+1}{2})_r}^\alpha$ respectively. Thus, this definition of the FSM is equivalent to the previous one, coming from fuzzy ordering and expressed in relation (4). If n is even, any couple of numbers that belong in the intervals $[x_{(\frac{n}{2})_l}^\alpha, x_{(\frac{n}{2}+1)_l}^\alpha]$ and $[x_{(\frac{n}{2})_r}^\alpha, x_{(\frac{n}{2}+1)_r}^\alpha]$ minimize (6) and can be considered as the median. In this case, suitable definitions can be given in order to meet the definitions coming from fuzzy ordering.

In contrast to the crisp case, where the scalar median of a set of numbers is always a number that belongs to the set, the FSM can be a different number, composed by the limits of the α-cuts of more than one fuzzy numbers. Definition 1 leads to this results when it is minimized unconditionally. It could lead us to an alternative definition under the constraint that the median should be one of the fuzzy numbers that belong to the set.

Definition 2 *The Fuzzy Scalar Median (FSM) of $X_1, X_2, \ldots, X_n$ fuzzy numbers is the fuzzy number X_m such that $X_m \in \{X_1, X_2, \ldots, X_n\}$ and $\forall j = 1, 2, \ldots, n,\ j \neq m$: $\sum_{i=1}^{n} D_n[X_m, X_i] < \sum_{i=1}^{n} D_n[X_j, X_i]$*

By using the second definition the fuzzy median is always a fuzzy number that belongs to the set. But, in contrast to the first definition where the most possible value (center) of the fuzzy median is always the median of the fuzzy numbers most possible values (centers), the center of the fuzzy median using the second definition, may be any of the fuzzy numbers centers. It depends on the fuzziness that every fuzzy number holds, in comparison to the fuzziness of the rest fuzzy numbers.

4 The Angle Decomposed Fuzzy Vectors (ADFV)

In the following, the Angle Decomposed Fuzzy Vectors (ADFV) will be defined as a subset of multidimensional fuzzy numbers and will provide us the ability to define a distance between them. Let $\mathbf{X}$ be an n-dimensional fuzzy set, $\mu_{\mathbf{X}}(\mathbf{x})$ be its membership function of n variables and $\mathbf{X}^{\alpha}$ the corresponding α-cuts, defined similarly with the 1-d case as the classical sets $\mathbf{X}^{\alpha}$, where $\mathbf{x} \in \mathbf{X}^{\alpha} \Leftrightarrow \mu(\mathbf{x}) \geq \alpha$, $\alpha \in [0, 1]$. Consider also that there is only one vector $\mathbf{x}_c$ where $\mu_{\mathbf{X}}(\mathbf{x}_c) = 1$, called in the following the center of the fuzzy set. Consider also $n-1$ angles $\theta = (\theta_i, i = 1, 2, \ldots, (n-1))$, $\theta_i \in [0, \pi)$. The center of the fuzzy set $\mathbf{x}_c$ and each angle θ_i determine a hyperplane. The intersection of $n-1$ hyperplanes is a straight line (direction) in the n-dimensional hyperspace, where a function μ_1 can be defined as $\mu_1(x, \theta) = \mu_X(x_1(x, \theta), x_2(x, \theta), \ldots, x_{n-1}(x, \theta), x)$. This function can be considered as a membership function of an 1-dimensional fuzzy set X^{θ}. Then, the ADFV are defined as follows:

Definition 3 *An n-dimensional fuzzy set $\mathbf{X}$ is an Angle Decomposed Fuzzy Vector (ADFV), if, for each vector of angles $\theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, the 1-d fuzzy set $X^{\theta} = \{x, \mu_1(x, \theta)\}$ is a convex fuzzy number.*

An example of a 2-dimensional ADFV and the angle decomposed 1-d convex fuzzy numbers is shown in Figure 3a. It is easy to prove that any ADFV is a fuzzy vector. We can also prove that if θ is a $(n-1-k)$-dimensional vector, the function $\mu_k(x_1, x_2, \ldots, x_k, \theta)$ can be considered as a membership function of a k-dimensional ADFV. The use of ADFV gives us the ability to establish a one to one correspondence between the points of two ADFV that limit their α-cuts on a certain direction.

By using this correspondence, a distance measure between multidimensional ADFV will be defined as an extension of the distance between 1-d fuzzy numbers given in (5) extended to any classical distance norm $||.,.||$ and to the n-dimensional ADFV as:

$$D_n[\mathbf{X}, \mathbf{Y}] = \frac{1}{2(n-1)\pi} \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{n-1}=0}^{\pi} \int_{\alpha=0}^{1} (||\mathbf{x}_l^{\theta\alpha}, \mathbf{y}_l^{\theta\alpha}|| + ||\mathbf{x}_r^{\theta\alpha}, \mathbf{y}_r^{\theta\alpha}||) d\alpha d\theta_{n-1} \ldots d\theta_1 \qquad (8)$$

where $\mathbf{x}_l^{\theta\alpha}$, $\mathbf{y}_l^{\theta\alpha}$ and $\mathbf{x}_r^{\theta\alpha}$, $\mathbf{y}_r^{\theta\alpha}$ are the lower and upper points that limit the α-cuts of the corresponding 1-d X^{θ} fuzzy numbers. In the following, the α-cuts of the X^{θ} fuzzy vectors will be called $\theta\alpha$-cuts and will be symbolized as $\mathbf{X}^{\theta\alpha}$. The use of ADFV guarantees that every point that belongs to the line segment from $\mathbf{x}_l^{\theta\alpha}$ to $\mathbf{x}_r^{\theta\alpha}$ belongs also to the $\theta\alpha$-cut. It is easy to be proven, by using the classical properties and (8), that the distance properties are still valid.

When the Euclidean norm is chosen to define a distance between two classical n-dimensional vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ as:

$$d_e^2(\mathbf{x}, \mathbf{y}) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2 \tag{9}$$

the Euclidean fuzzy distance can be defined by using (8) and (9). When the fuzzy vectors are described by using α-cuts, for a given α and a vector of angles $\theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, two points $\mathbf{x}_l^{\theta\alpha}$ and $\mathbf{x}_r^{\theta\alpha}$ are defined, which are the lower and the upper limits of the corresponding $\theta\alpha$-cut. The proposed Euclidean fuzzy distance is the normalized integral of all the distances $d_e^2(\mathbf{x}_l^{\theta\alpha}, \mathbf{y}_l^{\theta\alpha})$ between the lower limits, and the distances $d_e^2(\mathbf{x}_r^{\theta\alpha}, \mathbf{y}_r^{\theta\alpha})$ between the upper limits, for every $\alpha \in [0, 1]$ and $\theta_i \in [0, \pi)$, $i = 1, 2, \ldots, n-1$.

Let us symbolize as $d_{lx}^{\theta\alpha}$ the Euclidean distance between the lower limit $\mathbf{x}_l^{\theta\alpha}$ of the $\theta\alpha$-cut and the center $\mathbf{x}_c$ of an ADFV $\mathbf{X}$, as $d_{rx}^{\theta\alpha}$ the Euclidean distance between the upper limit $\mathbf{x}_r^{\theta\alpha}$ of the $\theta\alpha$-cut and the center $\mathbf{x}_c$ of an ADFV $\mathbf{X}$, and as d_{xy} the distance between the centers of two ADFV $\mathbf{X}, \mathbf{Y}$, as shown in Figure 3b. Then, the distances between two lower and two upper limits of two ADFV α-cuts are equal to:

$$d_e^2(\mathbf{x}_l^{\theta\alpha}, \mathbf{y}_l^{\theta\alpha}) = (d_{lx}^{\theta\alpha} - d_{ly}^{\theta\alpha})^2 + 2(d_{lx}^{\theta\alpha} - d_{ly}^{\theta\alpha}) d_{xy} \prod_{i=1}^{n-1} \cos(\theta_i) + d_{xy}^2 \tag{10}$$

$$d_e^2(\mathbf{x}_r^{\theta\alpha}, \mathbf{y}_r^{\theta\alpha}) = (d_{rx}^{\theta\alpha} - d_{ry}^{\theta\alpha})^2 - 2(d_{rx}^{\theta\alpha} - d_{ry}^{\theta\alpha}) d_{xy} \prod_{i=1}^{n-1} \cos(\theta_i) + d_{xy}^2 \tag{11}$$

where θ_i, $i = 1, 2, \ldots, n-1$ are known angles $\theta_i \in [0, \pi)$. By using (8),(10) and (11) the Euclidean fuzzy distance between two ADFV X, Y is given by $D_{e_n}[\mathbf{X}, \mathbf{Y}] = d_{xy}^2 + d_{f_{xy}}^2$ where:

$$\begin{aligned} d_{f_{xy}}^2 = \frac{1}{2(n-1)\pi} \int_{\theta_1=0}^{\pi} \cdots \int_{\theta_{n-1}=0}^{\pi} \int_{\alpha=0}^{1} & [(d_{lx}^{\theta\alpha} - d_{ly}^{\theta\alpha})^2 + (d_{rx}^{\theta\alpha} - d_{ry}^{\theta\alpha})^2 + \\ & + 2 d_{xy} \prod_{i=1}^{n-1} \cos(\theta_i) (d_{lx}^{\theta\alpha} - d_{ly}^{\theta\alpha} - d_{rx}^{\theta\alpha} + d_{ry}^{\theta\alpha})] d\alpha \, d\theta_{n-1} \ldots d\theta_1 \end{aligned} \tag{12}$$

The above equation shows that the proposed Euclidean fuzzy distance is the classical Euclidean distance between the centers of two ADFV $\mathbf{X}, \mathbf{Y}$, modified by a factor that depends on the fuzziness that every ADFV holds. The Euclidean fuzzy distance can be considered as a generalized Euclidean distance, since equation (12) becomes equal to 0 when the ADFV are crisp vectors ($d_{lx}^{\theta\alpha} = d_{ly}^{\theta\alpha} = d_{rx}^{\theta\alpha} = d_{ry}^{\theta\alpha} = 0,\ \forall\theta_i, \alpha$). Figure 4 shows an example of the distance between two 2-d ADFV, depending on their fuzziness.

5 The Fuzzy Vector Median (FVM) definition and properties

Based on the previously defined distance of ADFV, we extend the classical definition of the VM as follows:

Definition 4 *The Fuzzy Vector Median (FVM) of* $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ *ADFV is the ADFV* $\mathbf{X}_{FVM}$ *such that* $\mathbf{X}_{FVM} \in \{\mathbf{X}_i, i = 1, 2, \ldots, n\}$ *and* $\forall j = 1, 2, \ldots, n,\ j \neq k$*:* $\sum_{i=1}^{n} D_n[\mathbf{X}_{FVM}, \mathbf{X}_i] < \sum_{i=1}^{n} D_n[\mathbf{X}_j, \mathbf{X}_i]$

When the Euclidean fuzzy distance is used the Euclidean FVM is defined. Similarly to the classical VM, the Euclidean FVM $\mathbf{X}_{\mathrm{FVM}}$ does not minimize the unconditional expression:

$$\mathbf{S}_{e_i} = \sum_{i=1}^{n} D_{e_n}[\mathbf{X}_i, \mathbf{Y}] \tag{13}$$

but, by definition, it minimizes the same expression, when $\mathbf{Y}$ should be one of $\mathbf{X}_i$. The proposed distance of two fuzzy vectors depends on the distance of the crisp fuzzy set centers, and on the fuzziness that every fuzzy vector holds. As a result, the FVM of a set of fuzzy vectors is affected by the presence of fuzziness. The fuzzy vector, that its center is the classical VM of the fuzzy vector centers, may be substituted by another fuzzy vector when its fuzziness is different from the fuzziness of its neighbouring vectors.

In the following, the condition which should be valid for such a substitution will be found. Let us symbolize by d_{ij} the distance of the centers of two ADFV $\mathbf{X}_i, \mathbf{X}_j$ that can be calculated by (9), and $d_{f_{ij}}$ the distance depending on the fuzziness given by (12). The classical VM of the centers of the ADFV can be found by calculating the sums

$$S_{e_i} = \sum_{j=1}^{n} d_{ij} = \sum_{j=1}^{n} d_e^2(\mathbf{x}_{c_i}, \mathbf{x}_{c_j}) \tag{14}$$

Without loss of generality, we assume that $S_{e_i} < S_{e_{i+1}},\ \forall i = 1, 2, \ldots, n-1$, which means that $\mathbf{x}_{c_i}$ is the classical VM of the centers. Thus, by using (14), the following is also valid:

$$S_{e_{(i+k)}} - S_{e_i} = C_{(i+k)i} > 0 \quad k = 1, \ldots, n-i \tag{15}$$

The ADFV $\mathbf{X}_{i+k}$ will be the FVM if and only if $\mathbf{S}_{e_{(i+k)}} < \mathbf{S}_{e_j}$, $j = 1, 2, \ldots, n$, $j \neq i+k$. By using (15) the above condition is equivalent to:

$$\sum_{j=1}^{n} d_{f_{(i+k)j}} < \sum_{j=1}^{n} d_{f_{ij}} - C_{(i+k)i} \tag{16}$$

The above equation shows that the vector that corresponds to the classical Vector Median is the most probable candidate to be the FVM since $C_{ii} = 0$. It is also more probable to be substituted by its ordered neighbours, and the probability is reduced as the classical distance of the centers C increases.

6 Conclusions

Two alternative definitions of the Fuzzy Scalar Median were proposed, by using the ordering of fuzzy numbers based on the extensions of the minimum and maximum operations through the extension principle, and the minimization of a fuzzy distance measure, and the equivalence of the two definitions was proven. Then, the Fuzzy Vector Median was proposed, defined as an extension of Vector Median, based on a novel distance definition of fuzzy vectors and its basic properties were investigated.

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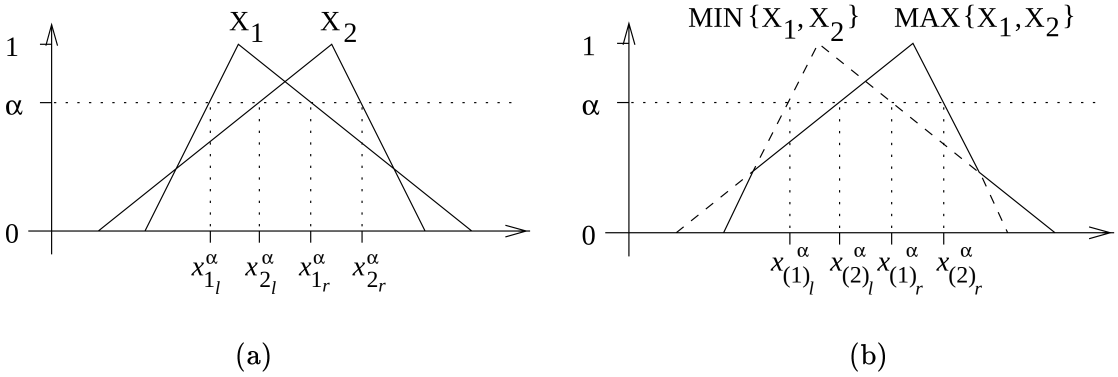
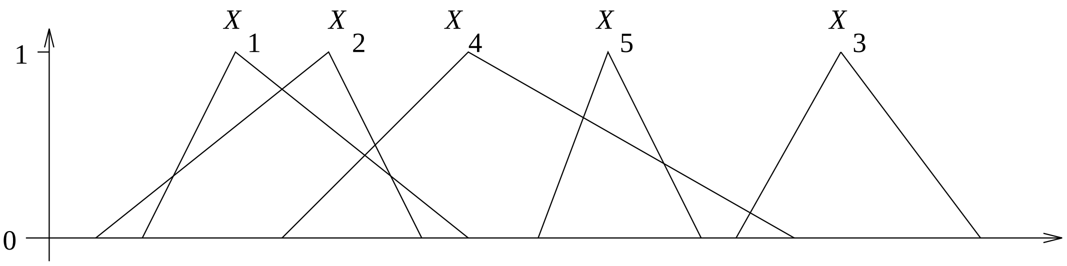
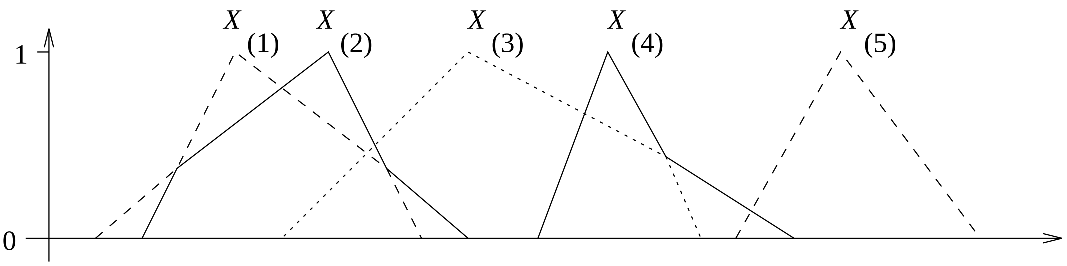


Figure 1: (a). Two fuzzy numbers X_1, X_2 and their α-cuts limits. (b). Maximum (solid line) and minimum (dashed line) of two fuzzy numbers.

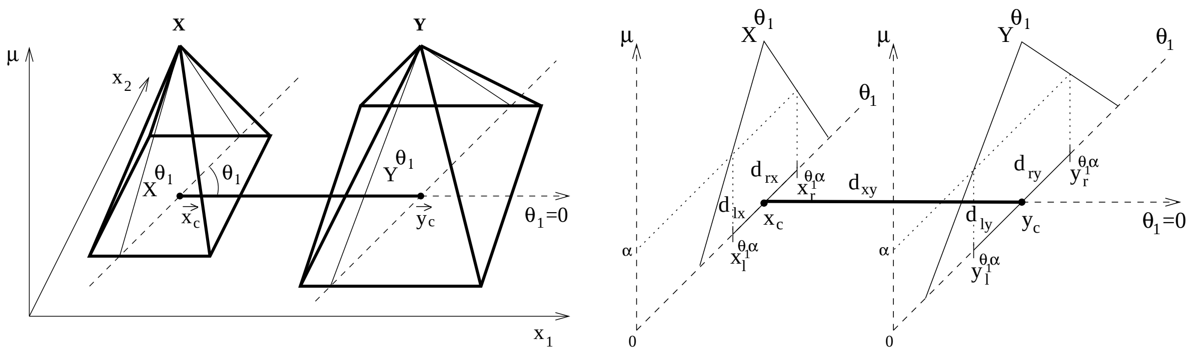


(a)



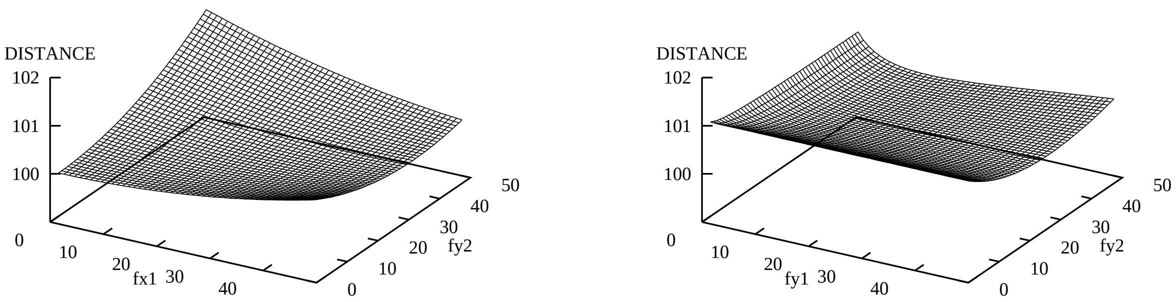
(b)

Figure 2: (a). Five fuzzy numbers X_1, X_2, X_3, X_4, X_5. (b). The fuzzy order statistics $X_{(1)}$, $X_{(2)}$, $X_{(3)}$, $X_{(4)}$, $X_{(5)}$.



(a) (b)

Figure 3: (a) Two 1-d convex fuzzy numbers X^{θ_1}, Y^{θ_1}, coming from two 2-d ADFV $\mathbf{X}$,$\mathbf{Y}$ when the angle vector $\theta = (\theta_1)$ is determined. (b) The upper $\mathbf{x}_r^{\theta_1\alpha}, \mathbf{y}_r^{\theta_1\alpha}$ and lower $\mathbf{x}_l^{\theta_1\alpha}, \mathbf{y}_l^{\theta_1\alpha}$ limits of two $\theta\alpha$-cuts $X^{\theta\alpha}, Y^{\theta\alpha}$, the centers of the ADFV x_c, y_c and the distances between them.



(a) (b)

Figure 4: The distance between two 2-d ADFV $\mathbf{X}, \mathbf{Y}$ with elliptical α-cuts and axes $f_{x1}^{\alpha}, f_{x2}^{\alpha}$ and $f_{y1}^{\alpha}, f_{y2}^{\alpha}$ respectively, which are reduced linearly from their maximum values $f_{x1}, f_{x2}, f_{y1}, f_{y2}$ for $\alpha = 0$, to zero for $\alpha = 1$. (a) the distance of the centers is 100, f_{x1}^{α} and f_{y2}^{α} vary from 0 to 50, and $f_{x2}^{\alpha} = \frac{f_{x1}^{\alpha}}{2}$, $f_{y1}^{\alpha} = \frac{f_{y2}^{\alpha}}{2}$. (b) the distance of the centers is 100, f_{y1}^{α} and f_{y2}^{α} vary from 0 to 50, and $f_{x1} = 10$, $f_{x2} = 30$.

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