Nonlinear Location and Scale Estimators of Fuzzy numbers

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Abstract

In this paper, the extension principle is used in order to fuzzify location and scale estimators when used on fuzzy numbers. First, fuzzy nonlinear means, such as arithmetic, harmonic, geometric and fuzzy L_p mean are defined as extensions of the corresponding crisp means. Fuzzy maximum and minimum operators are used to define order statistics of fuzzy numbers. Fuzzy L location and scale estimators, which are based on fuzzy order statistics, are defined as extensions of the crisp L location and scale estimators. The most widely used scale estimator, the sample standard deviation, is also extended to fuzzy numbers, through the extension principle. Equivalent relations that can be used to calculate the fuzzy estimators by using crisp arithmetic are also given, for each one of the proposed fuzzy estimators.

1 Introduction

The *fuzzy set theory* was first introduced by Zadeh [8]. He used this word to generalize the mathematical concept of the *set* to one of the *fuzzy set*. Suppose that the available information is such that the uncertain value can be located inside a closed interval $I \subset R$, which we call *interval of confidence*. Then, a *membership function* is defined that maps each element of the interval of confidence to a value in the interval $[0,1]$. The concept of a fuzzy number X is presented either by its membership function $X = \{(x, \mu_X(x)), x \in I_X\}$ or by the union of its $\alpha - cuts$ $X = \bigcup_\alpha \alpha \cdot [x_l^{(\alpha)}, x_r^{(\alpha)}]$, where $\alpha \in [0,1]$. In order to extend mathematical laws of crisp numbers in fuzzy theory we can use the *extension principle* [2, 3, 6], which provides the theoretical warranty that fuzzifies the parameters or arguments of a function resulting in computable fuzzy sets. The extensions of the basic operations are presented in Table 1.

The novel contribution of this paper, is the use of fuzziness concepts in estimation theory. The observation data will be considered to have uncertain values (fuzzy numbers). For example, fuzzy numbers can describe knowledge regarding the conditions of the observation, by changing fuzziness when observation conditions change. Fuzzy numbers can also be considered as the output of fuzzy inference mechanisms. When fuzzy numbers have to be combined in a nonlinear way, fuzzy nonlinear estimators (e.g. fuzzy median) have to be defined. In this paper, the extension principle is used to fuzzify location and scale estimators useful in estimation theory as well as in applications, e.g. in digital signal and image processing.

The paper has the following structure. In section 2 the fuzzy nonlinear means and the fuzzy L location estimators based on fuzzy order statistics are defined. In section 3 the fuzzy scale estimators, such as fuzzy L scale estimators and the fuzzy sample standard deviation, are also defined. Equivalent relations are also given for every estimator, which can be used to calculate the corresponding fuzzy estimators by using crisp arithmetic. Conclusions are drawn in section 4.

2 Location estimators of fuzzy numbers

The nonlinear means have been extensively used as location estimators of convolved and nonlinearly related data. Their definitions on crisp numbers can easily be extended to fuzzy numbers by using the extensions of the basic operations on fuzzy numbers, as they are presented in Table 1. The definitions of fuzzy arithmetic mean, fuzzy harmonic mean, fuzzy geometric mean and fuzzy L_p mean, as well as equivalent relations that can be used to calculate the corresponding fuzzy means by using crisp arithmetic, are given in Table 2.

In order to extend to fuzzy numbers, other location estimators that are based on order statistics, we shall first define the *fuzzy order statistics*. Let us assume that $X_1, X_2, \ldots, X_n$ are n fuzzy numbers. In order to rank them, we must successively use the fuzzy maximum MAX and minimum MIN operations as they are presented in Table 1. It can be easily shown that successive fuzzy MAX and MIN operations can be calculated by performing the corresponding successive crisp max, min operations on the crisp limits of the $\alpha - cuts$ of the fuzzy numbers. If we symbolize the fuzzy order statistics as $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where $X_{(1)}$ is the fuzzy minimum and $X_{(n)}$ the fuzzy maximum, they can be calculated by the following equations:

$$X_{(i)} = \bigcup_{\alpha} \alpha \cdot [x_{(i)_l}^{(\alpha)}, x_{(i)_r}^{(\alpha)}], \qquad i = 1, 2, \ldots, n \tag{1}$$

where $x_{(i)_l}^{(\alpha)}$ are the order statistics of $x_{i_l}^{(\alpha)}$ and $x_{(i)_r}^{(\alpha)}$ are the order statistics of $x_{i_r}^{(\alpha)}$, having the following ordering:

$$x_{(1)_l} \leq x_{(2)_l} \leq \ldots \leq x_{(n)_l} \tag{2}$$

$$x_{(1)_r} \leq x_{(2)_r} \leq \ldots \leq x_{(n)_r} \tag{3}$$

The fuzzy order statistics do not generally correspond to the input fuzzy numbers, as in crisp arithmetic. For example let X_1, X_2, X_3, X_4, X_5 be five fuzzy numbers illustrated in Figure 1a. Fuzzy order statistics $X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}, X_{(5)}$ are illustrated in Figure 1b. In this example only $X_{(5)}$ of the fuzzy order statistics is equal to X_3 of the input fuzzy numbers.

The L estimators of location are explicitly defined and easily calculated. Therefore, they have found extensive applications in digital signal and image processing. Their definitions [5], which are based on crisp order statistics, can easily be generalized to define fuzzy L location estimators. Since the fuzzy order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ of n fuzzy numbers can be easily computed as it was previously shown, the fuzzy L location estimators are defined as it is shown in Table 3, where w_i are crisp weights. Special cases of L estimators are the median, the midpoint and the a-trimmed mean, which are all based on order statistics. Their definitions on fuzzy numbers and equivalent relations, that can be used to calculate the corresponding fuzzy estimators by using crisp arithmetic, are also given in Table 3.

3 Scale estimators of fuzzy numbers

3.1 Fuzzy L estimators of scale

L-scale estimators of crisp numbers are based on order statistics of crisp numbers [5]. We shall generalize the definition of crisp L scale estimators, to define fuzzy L scale estimators. Let the observation data be n fuzzy numbers $X_1, X_2, \ldots, X_n$. The fuzzy order statistics $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ of the n fuzzy numbers can be easily computed as it was shown in the previous section. Then fuzzy L scale estimators are defined as:

$$T_n = \sum_{i=1}^{n} w_i X_{(i)} \tag{4}$$

where $X_{(i)}$ is the i-th fuzzy order statistic of the fuzzy data and w_i are appropriately chosen crisp weights. If $w_i = -1$ for $i = tn$, $w_i = 1$ for $i = (1-t)n$ and $w_i = 0$ otherwise, then the *fuzzy t-quantile range* is defined. Special cases of fuzzy t-quantile range are the *fuzzy interquantile range* for $t = 1/4$, and the *fuzzy quasi-range* ($W_{(i)}$) for $t \neq 0$ given by:

$$W_{(i)} = X_{(n+1-i)} - X_{(i)}, \qquad 2 \leq i \leq [\frac{n}{2}] \tag{5}$$

A special case of the fuzzy quasi-range for $i = 1$ is the *fuzzy range* (W) given by:

$$W = X_{(n)} - X_{(1)} \tag{6}$$

Another scale estimator is the *fuzzy thickened range* given by:

$$J_i = W + W_{(2)} + \ldots + W_{(i)} \tag{7}$$

3.2 Fuzzy sample standard deviation

Let us assume that x_i is the observation data and $\overline{x}$ is the arithmetic mean of x_i. Then, the crisp sample standard deviation (s) is defined by:

$$s = [\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2]^{\frac{1}{2}} \tag{8}$$

In order to extend it to *fuzzy sample standard deviation (S)*, the crisp functions of the square and the square root must be first extended to fuzzy numbers. Let X be a fuzzy number with its interval of confidence in R. We define the *positive fuzzy numbers* as the subset of fuzzy numbers with positive upper limits $x_r^{(\alpha)} \geq 0$ of the corresponding $\alpha - cuts$ for every $\alpha \in [0, 1]$.

The square X^2 of a fuzzy number X with its interval of confidence in R, can be defined as the product of two fuzzy numbers X_1, X_2 where $X_1 = X_2 = X$. Then by using the extension of multiplication given in Table 1 the square of a fuzzy number can be written as:

$$X^2 = \bigcup_{\alpha} \alpha \cdot [min\{(x_l^{(\alpha)})^2, x_l^{(\alpha)} \cdot x_r^{(\alpha)}, (x_r^{(\alpha)})^2\}, max\{(x_l^{(\alpha)})^2, (x_r^{(\alpha)})^2\}] \tag{9}$$

It is obvious from (9) that the square of a fuzzy number is always a positive fuzzy number but the lower limits of the corresponding $\alpha - cuts$ are not always positive.

The square root can be defined only for positive fuzzy numbers. Let $X^{\frac{1}{2}}$ be the square root of a fuzzy number X. It can be defined as the fuzzy number Y such as $Y^2 = X$. Then, by using (9) the square root of a fuzzy number is written as:

$$Y = X^{\frac{1}{2}} = \bigcup_{\alpha} \alpha \cdot [y_l^{(\alpha)}, y_r^{(\alpha)}] \tag{10}$$

where for every $\alpha \in [0,1]$:

$$y_r^{(\alpha)} = (x_r^{(\alpha)})^{\frac{1}{2}} \tag{11}$$

$$y_l^{(\alpha)} = \begin{cases} (x_l^{(\alpha)})^{\frac{1}{2}} & \text{if } x_l^{(\alpha)} \geq 0 \\ \\ \frac{x_l^{(\alpha)}}{y_r^{(\alpha)}} & \text{if } x_l^{(\alpha)} < 0 \end{cases} \tag{12}$$

It is supposed that the square roots of crisp numbers are the positive ones. Then, from (11) is obvious that the square root of a fuzzy number is a positive fuzzy number.

Let $X_1, X_2, \ldots, X_n$ be the fuzzy data and $\overline{X}$ be the arithmetic mean of X_i, $i = 1, \ldots, n$ given in Table 2. Then, fuzzy sample standard deviation S can be defined as an extension of crisp sample standard deviation:

$$S = [\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2]^{\frac{1}{2}} \tag{13}$$

The square root in (13) is always defined since the sum of positive fuzzy numbers is always a positive fuzzy number. By using Table 1 and (9-12), fuzzy sample standard deviation can be calculated by:

$$S = \bigcup_{\alpha} \alpha \cdot [s_l^{(\alpha)}, s_r^{(\alpha)}] \tag{14}$$

where for every $\alpha \in [0,1]$:

$$s_r^{(\alpha)} = [\frac{1}{n-1} \sum_{i=1}^{n} max\{(x_{i_l}^{(\alpha)} - \overline{x}_r^{(}(\alpha))^2, (x_{i_r}^{(\alpha)} - \overline{x}_l^{(\alpha)})^2\}]^{\frac{1}{2}} \tag{15}$$

$$s_l^{(\alpha)} = \begin{cases} z^{\frac{1}{2}} & \text{if } z \geq 0 \\ \\ \frac{z}{s_r^{(\alpha)}} & \text{if } z < 0 \end{cases} \tag{16}$$

where:

$$z = \frac{1}{n-1}\sum_{i=1}^{n} min\{(x_{i_l}^{(\alpha)} - \overline{x}_r^{(\alpha)})^2, (x_{i_l}^{(\alpha)} - \overline{x}_r^{(\alpha)}) \cdot (x_{i_r}^{(\alpha)} - \overline{x}_l^{(\alpha)}), (x_{i_r}^{(\alpha)} - \overline{x}_l^{(\alpha)})^2\} \quad (17)$$

It is obvious from (15) that the fuzzy sample standard deviation is always a positive fuzzy number.

4 Conclusions

The crisp nonlinear estimators have been extensively used as location and scale estimators of convolved and nonlinearly related data. The need for modeling uncertainty in the observation data, e.g. knowledge about the observation conditions, necessitates the substitution of crisp numbers by fuzzy numbers in many cases. The use of crisp nonlinear estimators on defuzzified fuzzy data would possibly result the loss of the information contained in fuzziness. By using fuzzy nonlinear estimators the information that fuzziness holds is transferred to the estimated fuzzy number.

We have extended the nonlinear means that are widely used as location estimators. We have also presented the fuzzy order statistics and we have defined fuzzy location and scale estimators based on them. Finally, fuzzy sample standard deviation, the most widely used scale estimator, was also defined. For each one of the proposed fuzzy estimators, equivalent relations that can be used to calculate them by using crisp arithmetic, was given.

References

[1] J. C. Bezdek, S. K. Pal. *"Fuzzy Models for Pattern Recognition"*, IEEE Press, New York, 1992.

[2] D. Dubois, H. Prade. *"Fuzzy Sets and Systems, Theory and Applications"*, Academic Press, Boston, 1980.

[3] A. Kaufmann, M. M. Gupta. *"Introduction to Fuzzy Arithmetic: Theory and Applications"*, Van Nostrand Reinhold, New York, 1985.

[4] R. Kruse, K. D. Meyer. *"Statistics with Vague Data"*, D.Reidel Publishing Company, Dordrecht, 1987.

[5] I. Pitas, A. N. Venetsanopoulos. *"Nonlinear Digital Filters"*, Kluwer Academic Publishers, Boston, 1990.

[6] L. H. Tsoukalas, R. E. Uhrig. *"Fuzzy and Neural Approaches in Engineering"*, 1993.

[7] S. G. Tzafestas, A. N. Venetsanopoulos. *"Fuzzy Reasoning in Information, Decision and Control Systems"*, Kluwer Academic Publishers, Dordrecht, 1994.

[8] L. Zadeh. *"Fuzzy Sets"*, IEEE Inform. and Control, vol 8, no 3, pp.338-353, 1965.

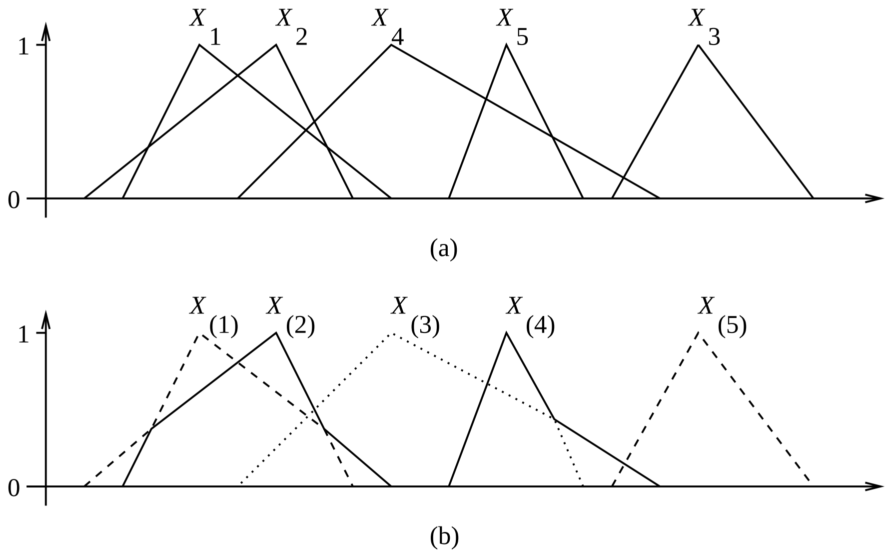

Figure 1: (a). Five fuzzy numbers X_1, X_2, X_3, X_4, X_5. (b). The fuzzy order statistics $X_{(1)}$, $X_{(2)}$, $X_{(3)}$, $X_{(4)}$, $X_{(5)}$.

Table 1: Basic operations of fuzzy numbers, extended through the extension principle [2,3,6].

Operation	**Crisp**	**Fuzzy**
Addition	$x_1 + x_2$	$X_1 + X_2 = \bigcup_{\alpha} \alpha \cdot [x_{1_l}^{(\alpha)} + x_{2_l}^{(\alpha)}, x_{1_r}^{(\alpha)} + x_{2_r}^{(\alpha)}], \quad I_{X_1}, I_{X_2} \subset R$
Subtraction	$x_1 - x_2$	$X_1 - X_2 = \bigcup_{\alpha} \alpha \cdot [x_{1_l}^{(\alpha)} - x_{2_r}^{(\alpha)}, x_{1_r}^{(\alpha)} - x_{2_l}^{(\alpha)}], \quad I_{X_1}, I_{X_2} \subset R$
Multiplication	$x_1 \cdot x_2$	$X_1 \cdot X_2 = \bigcup_{\alpha} \alpha \cdot [min\{x_{1_l}^{(\alpha)} \cdot x_{2_l}^{(\alpha)}, x_{1_l}^{(\alpha)} \cdot x_{2_r}^{(\alpha)}, x_{1_r}^{(\alpha)} \cdot x_{2_l}^{(\alpha)}, x_{1_r}^{(\alpha)} \cdot x_{2_r}^{(\alpha)}\},$ $max\{x_{1_l}^{(\alpha)} \cdot x_{2_l}^{(\alpha)}, x_{1_l}^{(\alpha)} \cdot x_{2_r}^{(\alpha)}, x_{1_r}^{(\alpha)} \cdot x_{2_l}^{(\alpha)}, x_{1_r}^{(\alpha)} \cdot x_{2_r}^{(\alpha)}\}], \quad I_{X_1}, I_{X_2} \subset R$
Inverse	x^{-1}	$X^{-1} = \bigcup_{\alpha} \alpha \cdot [\frac{1}{x_r^{(\alpha)}}, \frac{1}{x_l^{(\alpha)}}], \quad I_X \subset R^+$
Maximum	$max\{x_1, x_2\}$	$MAX\{X_1, X_2\} = \bigcup_{\alpha} \alpha \cdot [max\{x_{1_l}^{(\alpha)}, x_{2_l}^{(\alpha)}\}, max\{x_{1_r}^{(\alpha)}, x_{2_r}^{(\alpha)}\}], \quad I_{X_1}, I_{X_2} \subset R$
Minimum	$min\{x_1, x_2\}$	$MIN\{X_1, X_2\} = \bigcup_{\alpha} \alpha \cdot [min\{x_{1_l}^{(\alpha)}, x_{2_l}^{(\alpha)}\}, min\{x_{1_r}^{(\alpha)}, x_{2_r}^{(\alpha)}\}], \quad I_{X_1}, I_{X_2} \subset R$

Table 2: Fuzzy nonlinear means definitions and equivalent relations for their calculation by crisp arithmetic, ($X_i, i=1,\ldots,n$: fuzzy data, w_i: crisp weights).

Estimator	Fuzzy Definition	Equivalent Relation
Arithmetic mean	$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$	$\overline{X} = \bigcup_{\alpha} \alpha \cdot [\frac{1}{n}\sum_{i=1}^{n} x_{i_l}^{(\alpha)}, \frac{1}{n}\sum_{i=1}^{n} x_{i_r}^{(\alpha)}]$
Harmonic mean	$Y_H = \frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} \frac{w_i}{X_i}}$	$Y_H = \bigcup_{\alpha} \alpha \cdot [\frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} \frac{w_i}{x_{i_l}^{(\alpha)}}}, \frac{\sum_{i=1}^{n} w_i}{\sum_{i=1}^{n} \frac{w_i}{x_{i_r}^{(\alpha)}}}]$
Geometric mean	$Y_G = \log^{-1}(\frac{\sum_{i=1}^{n} w_i \log X_i}{\sum_{i=1}^{n} w_i})$	$Y_G = \bigcup_{\alpha} \alpha \cdot [\log^{-1}(\frac{\sum_{i=1}^{n} w_i \log x_{i_l}^{(\alpha)}}{\sum_{i=1}^{n} w_i}), \log^{-1}(\frac{\sum_{i=1}^{n} w_i \log x_{i_r}^{(\alpha)}}{\sum_{i=1}^{n} w_i})]$
L_p mean	$Y_{L_p} = (\frac{\sum_{i=1}^{n} w_i X_i^p}{\sum_{i=1}^{n} w_i})^{\frac{1}{p}}$	$Y_{L_p} = \bigcup_{\alpha} \alpha \cdot [(\frac{\sum_{i=1}^{n} w_i (x_{i_l}^{(\alpha)})^p}{\sum_{i=1}^{n} w_i})^{\frac{1}{p}}, (\frac{\sum_{i=1}^{n} w_i (x_{i_r}^{(\alpha)})^p}{\sum_{i=1}^{n} w_i})^{\frac{1}{p}}]$

Table 3: Fuzzy L location estimators definitions and equivalent relations for their calculation by crisp arithmetic, ($X_i, i=1,\ldots,n$: fuzzy data, $X_{(i)}, i=1,\ldots,n$: fuzzy order statistics, w_i: crisp weights).

Estimator	Fuzzy Definition	Equivalent Relation
L estimator	$T_n = \sum_{i=1}^{n} w_i X_{(i)}$	$T_n = \bigcup_{\alpha} \alpha \cdot [\sum_{i=1}^{n} \min\{w_i x_{(i)_l}^{(\alpha)}, w_i x_{(i)_r}^{(\alpha)}\}, \sum_{i=1}^{n} \max\{w_i x_{(i)_l}^{(\alpha)}, w_i x_{(i)_r}^{(\alpha)}\}]$
a-trimmed mean	$\overline{X}_a = \frac{1}{n(1-2a)} \sum_{i=an+1}^{n-an} X_{(i)}, \quad a < 0.5$	$\overline{X}_a = \bigcup_{\alpha} \alpha \cdot [\frac{1}{n(1-2a)} \sum_{i=an+1}^{n-an} x_{(i)_l}^{(\alpha)}, \frac{1}{n(1-2a)} \sum_{i=an+1}^{n-an} x_{(i)_r}^{(\alpha)}]$
Midpoint	$MIDP\{X_i\} = \frac{1}{2}(X_{(1)} + X_{(n)})$	$MIDP\{X_i\} = \bigcup_{\alpha} \alpha \cdot [\frac{1}{2}(x_{(1)_l}^{(\alpha)} + x_{(n)_l}^{(\alpha)}), \frac{1}{2}(x_{(1)_r}^{(\alpha)} + x_{(n)_r}^{(\alpha)})]$
Median	$MED\{X_i\} = \begin{cases} X_{(\nu+1)} \text{ if } n \text{ odd}, \\ \quad n = 2\nu+1 \\ \frac{1}{2}(X_{(\nu)} + X_{(\nu+1)}) \\ \quad \text{if } n \text{ even}, n = 2\nu \end{cases}$	$MED\{X_i\} = \begin{cases} \bigcup_{\alpha} \alpha \cdot [x_{(\nu+1)_l}^{(\alpha)}, x_{(\nu+1)_r}^{(\alpha)}] \\ \quad \text{if } n \text{ odd}, n = 2\nu+1 \\ \bigcup_{\alpha} \alpha \cdot [\frac{1}{2}(x_{(\nu)_l}^{(\alpha)} + x_{(\nu+1)_l}^{(\alpha)}), \frac{1}{2}(x_{(\nu)_r}^{(\alpha)} + x_{(\nu+1)_r}^{(\alpha)})] \\ \quad \text{if } n \text{ even}, n = 2\nu \end{cases}$