Optimal L-filters for vector magnitude filtering

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Abstract. Vector magnitude filtering on vectors represented in polar coordinates can be useful in cases where vector direction is either of no importance or must be preserved intact. This paper focuses on the design of optimal L-filters for the vector magnitude filtering of certain types of two-channel vector fields (rotational, translational and zooming motion vector fields) corrupted by additive noise. Expressions for the unconstrained, unbiased and location invariant L-filter coefficients are derived. Experiments conducted on artificially generated vector fields prove the superior performance of the proposed filters.

1. Introduction

Vector field processing is an important sub-task in many signal processing applications including image sequence processing and stereoscopic image processing. In the case of image sequences, motion vector fields are used in motion compensated coding and compression while disparity vector fields derived from stereo image pairs can be utilized in disparity compensated coding and also provide us with scene depth information. Vector field processing is also closely related to multichannel signal processing (e.g. color image processing) since multichannel observations are essentially vector-valued observations.

So far, vector fields were represented and treated in a cartesian coordinate system. However, vector representation in polar coordinates (i.e. representation in terms of vector magnitude and vector direction) might be preferable in certain situations [1, 2]. Filtering of vector magnitude without affecting its direction (e.g. in estimating wind velocity or car speed from noisy data) is such a case. It is obvious that this task cannot be performed successfully using filtering in cartesian coordinates. Furthermore, transforming certain types of vector fields (e.g. rotational vector fields) in polar coordinates can result in vectors exhibiting interesting properties, like the constant magnitude property. This paper deals with the design of optimal (with respect to the mean square error) L-filters [3] for the filtering of the vector magnitude of two-channel vector fields corrupted by additive bivariate noise. Unconstrained as well as unbiased and location invariant L-filters are studied.

2. Derivation of the optimal filter coefficients

2-D velocity vector fields corresponding to certain types of object motion can be handled more efficiently if transformed to polar coordinates. Such a type of motion is the constant velocity rotation around a fixed point (Figure 1). In this case, the velocity vectors are of constant magnitude on concentric circles around the center of rotation and have the same direction θ along each radiant (in fact they are perpendicular to the radiant). A similar case occurs when a camera zooms on a still object, or when an object moves towards a fixed camera.

It is well known [3] that the design of single-channel L-filters for the filtering of additive noise requires the signal to be constant. In the sequel, we shall show that this is also true in the design of L-filters for the filtering of vector magnitude when vectors are corrupted by bivariate additive noise. In the case of the above-mentioned vector field types, vector representation in terms of the polar coordinates (r, θ) exhibit the constant vector magnitude property when we are moving over certain paths (concentric circles around the center of rotation in the case of rotational fields). Therefore, we can exploit this property to design optimal L-filters. Another type of vector fields that possesses the constant signal property and therefore can be filtered using L-filters is constant velocity translational vector fields. We will present L-filters for this type of vector fields along with filters for rotational fields.

Let us suppose (Figure 1) that the observed two-dimensional vector signal in the case of the constant velocity rotational motion

$$\mathbf{w}(x,y) = [w_x(x,y), w_y(x,y)]^T \tag{1}$$

can be expressed as a sum of a signal vector

$$\mathbf{s}(x,y) = [s_x(x,y), s_y(x,y)]^T \tag{2}$$

and a zero mean noise vector

$$\mathbf{n}(x,y) = [n_x(x,y), n_y(x,y)]^T \tag{3}$$

whose components are distributed according to some bivariate distribution $f_{n_x,n_y}(n_x, n_y)$. The signal vector s can be expressed in polar coordinates as (r, θ) :

$$r = \sqrt{\mathbf{s}^T\mathbf{s}} = \sqrt{{s_x}^2 + {s_y}^2},\ \theta = arctan\frac{s_y}{s_x} \tag{4}$$

r is constant on circles around the center of rotation. Since the vectors in each point are perpendicular to the radiant from the center of rotation to this point, θ also defines the position of the vector on the circle. The squared magnitude u of $\mathbf{w}$ can be expressed as follows :

$$\begin{aligned} u = |\mathbf{w}|^2 = (n_x + s_x)^2 + (n_y + s_y)^2 = \\ n_x^2 + n_y^2 + r^2 + 2n_x r\cos\theta + 2n_y r\sin\theta = \\ \mathbf{n}^T\mathbf{n} + r^2 + 2r\mathbf{n}^T\mathbf{c} \end{aligned} \tag{5}$$

where :

$$\mathbf{c} = [\cos\theta, \sin\theta]^T$$

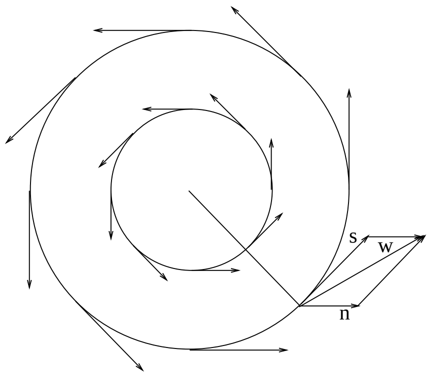

Figure 1. Constant velocity rotational motion field.

The output of an L-filter of length N operating on the squared magnitude sequence is :

$$y_i = \sum_{j=1}^{N} a_j u_{(j)} = \mathbf{a}^T\mathbf{R} = \mathbf{a}^T(r^2\mathbf{e} + \mathbf{N}) \tag{6}$$

where $\mathbf{e} = [1, 1 \ldots, 1]^T$, $\mathbf{R}, \mathbf{N}$ are the vectors of the ordered samples for u and $\nu = \mathbf{n}^T\mathbf{n} + 2\mathbf{n}^T\mathbf{c}r$ respectively, $u_{(j)}$ are the ordered samples $u_{i-k}, \ldots, u_i, \ldots, u_{i+k}$ ($N = 2k+1$) and $\mathbf{a}$ the filter coefficients vector. The MSE d between the filter output and the constant signal r^2 can be expressed in the following form :

$$d = r^4(\mathbf{a}^T\mathbf{e} - 1)^2 + 2r^2(\mathbf{a}^T\mathbf{e} - 1)\mathbf{a}^T\mu_{\mathbf{N}} + \mathbf{a}^T\mathbf{R}_{\mathbf{N}}\mathbf{a} \tag{7}$$

where $\mathbf{R}_{\mathbf{N}}$ is the correlation matrix of the ordered vector $\mathbf{N}$ and $\mu_{\mathbf{N}} = E[\mathbf{N}]$. The coefficients vector $\mathbf{a}$ that minimizes the MSE can be found by differentiating d with respect to the coefficients and equating the outcome to zero. By doing so, the following set of equations results :

$$\mathbf{A}\mathbf{a} - \mathbf{b} = 0 \tag{8}$$

where :

$$\mathbf{A} = 2r^4\mathbf{U} + 2\mathbf{R}_{\mathbf{N}} + 2r^2(\mu_{\mathbf{N}}\mathbf{e}^T + (\mu_{\mathbf{N}}\mathbf{e}^T)^T) \tag{9}$$

$$\mathbf{b} = 2r^4\mathbf{e} + 2r^2\mu_{\mathbf{N}} \tag{10}$$

In the previous expressions $\mathbf{U}$ is a square matrix having all elements equal to unity. By solving the set of equations (8) we find the optimum filter coefficients vector:

$$\mathbf{a} = \mathbf{A}^{-1}\mathbf{b} \tag{11}$$

The above coefficients correspond to the unconstrained L-filter. Structural constraints such as unbiasedness and location invariance can also be incorporated in the design of the L-filters. An L-filter is said to be *unbiased* estimator of location if the expected value of its output equals the original constant signal [3]. In our case, this constraint takes the following form:

$$E[y] = \mathbf{a}^T(r^2\mathbf{e} + \mu_{\mathbf{N}}) = r^2 \tag{12}$$

The MSE minimization problem becomes a constrained one and can be solved using the method of Lagrange multipliers. The coefficients vector that minimizes (7) subject to the constraint (12) is the one that minimizes the expression :

$$r^4(\mathbf{a}^T\mathbf{e} - 1)^2 + 2r^2(\mathbf{a}^T\mathbf{e} - 1)\mathbf{a}^T\mu_{\mathbf{N}} + \mathbf{a}^T\mathbf{R}_{\mathbf{N}}\mathbf{a} + \lambda(\mathbf{a}^T(r^2\mathbf{e} + \mu_{\mathbf{N}}) - r^2) \tag{13}$$

where λ is the Lagrange multiplier. By differentiating (13) with respect to $a_1, \ldots, a_N, \lambda$, equating the result to zero and then solving for $\mathbf{a}$ we come to the following expression :

$$\mathbf{a} = \mathbf{A}^{-1}\left(\mathbf{b} - \frac{\mathbf{b}^T(\mathbf{A}^{-1})^T(r^2\mathbf{e} + \mu_{\mathbf{N}}) - r^2}{(r^2\mathbf{e} + \mu_{\mathbf{N}})^T(\mathbf{A}^{-1})^T(r^2\mathbf{e} + \mu_{\mathbf{N}})}(r^2\mathbf{e} + \mu_{\mathbf{N}})\right) \tag{14}$$

An L-filter whose sum of coefficients equals unity i.e. $\mathbf{a}^T\mathbf{e} = 1$ is said to be *location invariant*. The coefficients vector that minimizes MSE subject to the previous constraint can also be found by applying the method of Lagrange multipliers and is given by the following expression :

$$\mathbf{a} = \mathbf{A}^{-1}\left(\mathbf{b} - \frac{\mathbf{e}^T\mathbf{A}^{-1}\mathbf{b} - 1}{\mathbf{e}^T\mathbf{A}^{-1}\mathbf{e}}\mathbf{e}\right) \tag{15}$$

In order to proceed with the filter design, the correlation matrix $\mathbf{R}_{\mathbf{N}}$ and the mean vector $\mu_{\mathbf{N}}$ for the order statistics vector $\mathbf{N}$ have to be evaluated using the expressions for the joint pdf of two order statistics [4]. The evaluation of the moments that appear as elements of $\mathbf{R}_{\mathbf{N}}$,$\mu_{\mathbf{N}}$ was done in a discrete way by quantizing the two-dimensional vectors of continuous random variables. The pdf $f_\nu(\nu)$ of the random variable ν must be derived for this calculation. The angle θ, i.e. the position on the circle where the filtering is performed is involved in these calculations. Therefore, different filter coefficients must be evaluated for each position around the circle, resulting in considerably large computational complexity. To overcome this, two different approaches were tested. First, it was assumed that θ is constant when moving within a small arc on the circle. The coefficients will have to be evaluated for every one of these arcs, in order to apply the filter on the entire circle. The computation of $f_\nu(\nu)$ under this assumption is done in two stages. First, we use an auxiliary variable $h = n_x$ and evaluate the joint pdf of ν, h, using the formula for the evaluation of the pdf of functions of random variables [5] :

$$f_{\nu,h}(\nu,h) = \frac{f_{n_x,n_y}(h, -rsin\theta + \sqrt{r^2+\nu-(h+rcos\theta)^2})}{2\sqrt{r^2+\nu-(h+rcos\theta)^2}} + \frac{f_{n_x,n_y}(h, -rsin\theta - \sqrt{r^2+\nu-(h+rcos\theta)^2})}{2\sqrt{r^2+\nu-(h+rcos\theta)^2}}$$
$$\nu > -r^2,\ g_1(\nu,\theta) < h < g_2(\nu,\theta) \tag{16}$$

where :

$$g_1(\nu,\theta) = -rcos\theta - \sqrt{r^2+\nu} \tag{17}$$

$$g_2(\nu,\theta) = -rcos\theta + \sqrt{r^2+\nu} \tag{18}$$

Note that $f_{\nu,h}(\nu,h)$ is zero outside the above-mentioned interval for ν, h. Then, $f_\nu(\nu)$ is calculated by integrating (16) with respect to h :

$$f_\nu(\nu) = \int_{g_1(\nu,\theta)}^{g_2(\nu,\theta)} \left(\frac{f_{n_x,n_y}(h, -rsin\theta + \sqrt{r^2+\nu-(h+rcos\theta)^2})}{2\sqrt{r^2+\nu-(h+rcos\theta)^2}} + \frac{f_{n_x,n_y}(h, -rsin\theta - \sqrt{r^2+\nu-(h+rcos\theta)^2})}{2\sqrt{r^2+\nu-(h+rcos\theta)^2}}\right)dh$$
$$\nu > -r^2 \tag{19}$$

The integrals in (19) were calculated using numerical integration.

Filtering the rotational field on a small arc over the circle, where θ is assumed to be constant, can be considered as equivalent to the filtering of a constant velocity translational vector field. Thus, the coefficients derived under this model can be used for the filtering of the vector magnitude of such a translational vector field, provided that the vector direction θ is known.

The second approach was to consider θ as being uniformly distributed around the circle. Then the filter coefficients will be the same for every filtering position around the circle. In this case, another assumption concerning the joint pdf of the three random variables n_x, n_y, θ has also to be made. The following model has been adopted :

$$f_{n_x,n_y,\theta}(n_x, n_y, \theta) = f_{n_x,n_y}(n_x, n_y) f_\theta(\theta) \tag{20}$$

where

$$f_\theta(\theta) = \frac{1}{2\pi},\ 0 \leq \theta < 2\pi \tag{21}$$

This is a realistic model, since the noise components $n_x,\ n_y$ do not have any dependency from the angle θ that defines the position on the circle. Again the evaluation of f_ν is done in two steps. We use an auxiliary random variable $h = n_x$, evaluate the joint pdf of ν, h, θ and integrate this expression with respect to h, θ to obtain $f_\nu(\nu)$:

$$\begin{gathered} f_\nu(\nu) = \\ \int_0^{2\pi} \int_{g_1(\nu,\theta)}^{g_2(\nu,\theta)} \Big(\frac{f_{n_x,n_y}(h, -r sin\theta + \sqrt{r^2+\nu-(h+rcos\theta)^2}) f_\theta(\theta)}{2\sqrt{r^2+\nu-(h+rcos\theta)^2}} + \\ \frac{f_{n_x,n_y}(h, -r sin\theta - \sqrt{r^2+\nu-(h+rcos\theta)^2}) f_\theta(\theta)}{2\sqrt{r^2+\nu-(h+rcos\theta)^2}} \Big) dh d\theta \\ \nu > -r^2 \end{gathered} \tag{22}$$

Using these pdfs we can calculate the correlation matrix $\mathbf{R_N}$ and the mean vector $\mu_{\mathbf{N}}$ whose elements appear in (11),(14),(15) and obtain the optimum unconstrained, unbiased or location invariant L-filter coefficients.

A practical consideration about the L-filters that have been presented is that the constant vector magnitude r must be known. In fact r is not known, actually this is what we are trying to estimate, and, therefore, we must use an initial estimate. For example, we can filter the vector field with a 2-D mean or marginal median filter, obtain an estimate for r and then use it to design an L-filter that will give a better estimate for the vector magnitude.

3. Experimental results

The performance of the L-filters that have been designed in the previous section has been evaluated using simulations and their noise suppression ability has been compared to that of other filters. Three sets of experiments have been conducted to test the performance of the L-filters designed both for constant direction angle θ, (i.e. for filters that can be used for filtering of rotational vector fields in a small arc over a circle around the center of rotation) and for uniformly distributed θ. In the first set of experiments, samples from a constant signal $r = 1$, $\theta = 0.5\ rad$ or $s_x = 0.877$, $s_y = 0.479$ in cartesian coordinates were corrupted by additive zero mean bivariate Gaussian noise

$$N(0, 0, \sigma_x, \sigma_y, \rho),\ \sigma_x = \sigma_y = 1,\ \rho = 0.5 \tag{23}$$

acting on the $x,\ y$ components of the signal. L-filters of length $N = 3, 5, 7$ were designed for this kind of noise, assuming $r,\ \theta$ to be known, and their performance has been compared to the performance of the following single channel filters performing on the squared vector magnitude: arithmetic mean, median and minimum filter. The performance of two-channel arithmetic mean and marginal median filters performing on the components $w_x,\ w_y$ of the noisy signal has also been evaluated. Since our final aim was to filter the vector magnitude r, the vector output of the two-channel filters was transformed to polar coordinates. The quantitative performance criterion that was used for the comparisons was the Noise Reduction Index

$$NRI = 10 log \frac{\sum_i (y_i - r^2)^2}{\sum_i (u_i - r^2)^2} \tag{24}$$

Comparative results are presented in Table 1. It can be seen that the unconstrained L-filter has the best performance among all filters, the unbiased L-filter being the second best. Both filters perform far more better than the rest. It is interesting to note that the minimum filter has the best performance among the other filters. The superiority of the L-filters is further illustrated in Figures 2-5. Figure 2 shows the vector magnitude of the noisy vector sequence described in the previous experiment. Figures 3, 4 and 5 show the same sequence filtered with minimum filter, arithmetic mean filter operating on $w_x,\ w_y$ components and unbiased L-filter respectively. All filters were of length $N = 5$. It is obvious that L-filter is much more better than the two other filters.

In the second set of experiments we designed and tested L-filters under the constant direction angle θ assumption for the suppression of bivariate Gaussian noise contaminated with uniform noise :

$$\begin{gathered} \epsilon N(0, 0, \sigma_x, \sigma_y, \rho) + (1-\epsilon) U([0, 1.5], [0, 1.5]), \\ \sigma_x = \sigma_y = 1,\ \rho = 0.5,\ \epsilon = 0.8 \end{gathered} \tag{25}$$

where $U([0, 1.5], [0, 1.5])$ denotes the pdf of uniformly distributed outliers in the domain $[0, 1.5] \times [0, 1.5]$. This type of noise was used to corrupt the same constant signal that was described in the previous simulation. Results can be seen in Table 2. The superiority of the L-filters is obvious.

The last set of experiments aimed at testing the performance of the L-filters designed under the hypothesis of θ being uniformly distributed in the interval $[0,\ 2\pi]$. The coefficients found under this approach can be used to perform filtering of the vector magnitude on the entire circle. In order to test the validity of this hypothesis we used a vector sequence whose vectors were of constant magnitude $r = 1$ while θ changed from 0 to 2π as we moved from the first to the last sequence point, i.e. vectors corresponding to a circular path around the center of rotation in a rotational vector field. This signal was corrupted by bivariate Gaussian noise (23). Results are tabulated in Table 3. Once again L-filters are superior than the rest of the filters and their performance is similar to the performance of the filters designed under the constant θ hypothesis. This can be considered as a proof that the proposed model fits very well the real situation.

Filter	N=3	N=5	N=7
Unconstrained L	<u>-17.01</u>	<u>-18.67</u>	<u>-19.40</u>
Unbiased L	-15.21	-18.25	-18.45
Location Invariant L	-12.04	-15.68	-17.78
Minumum on magnitude	-12.05	-14.55	-14.37
Mean on magnitude	-2.91	-3.85	-4.33
Median on magnitude	-4.27	-6.39	-7.54
Marginal median	-5.62	-8.63	-10.67
Mean	-7.26	-10.09	-11.93

Table 1. Performance of the various filters in the filtering of bivariate Gaussian noise from a vector sequence of constant magnitude and direction.

4. Conclusions

Optimal L-filters for the filtering of vector magnitude in certain types of vector fields have been proposed. Unconstrained as well as unbiased and location invariant L-filters have been designed. Experimental results verify the superiority of L-filters compared to other already known filters.

Filter	N=3	N=5	N=7
Unconstrained L	-18.12	-19.93	-20.63
Unbiased L	-16.75	-19.00	-19.92
Location Invariant L	-10.31	-14.62	-16.97
Minumum on magitude	-10.29	-13.94	-14.51
Mean on magnitude	-2.43	-3.14	-3.49
Median on magnitude	-3.28	-4.46	-5.22
Marginal median	-4.21	-6.12	-7.28
Mean	-5.68	-8.03	-9.42

Table 2. Performance of the various filters in the filtering of bivariate Gaussian noise contaminated with uniformly distributed outliers from a vector sequence of constant magnitude and direction.

Filter	N=3	N=5	N=7
Unconstrained L	-17.24	-19.08	-20.39
Unbiased L	-15.63	-17.95	-19.48
Location Invariant L	-11.47	-15.70	-17.87
Minumum on magnitude	-11.48	-14.54	-14.59
Mean on magnitude	-2.73	-3.59	-4.04
Median on magnitude	-4.03	-5.80	-6.83
Marginal median	-6.02	-9.68	-12.69
Mean	-7.94	-11.76	14.52

Table 3. Performance of the various filters in the filtering of bivariate Gaussian noise from a vector sequence having constant magnitude and direction that varies uniformly from 0 rad to 2π rad.

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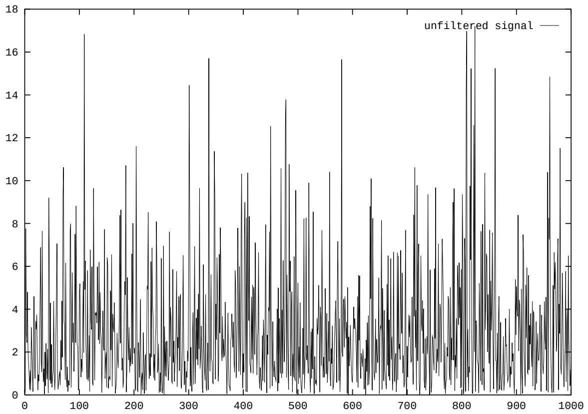

Figure 2. Vector magnitude of a noisy vector sequence.

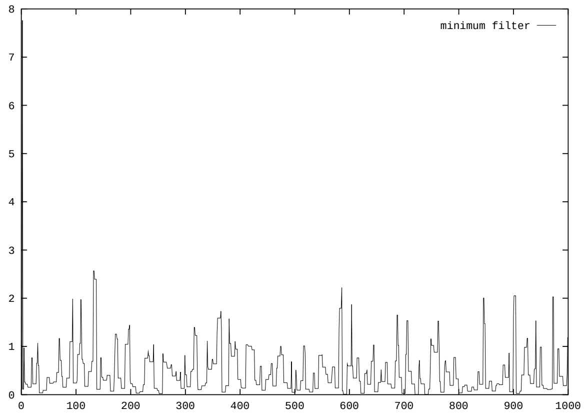

Figure 3. Vector magnitude sequence filtered using a minimum filter.

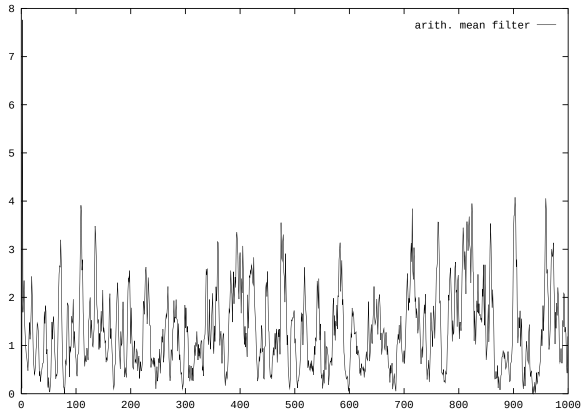

Figure 4. Vector magnitude of the noisy vector sequence filtered using two channel arithmetic mean filter.

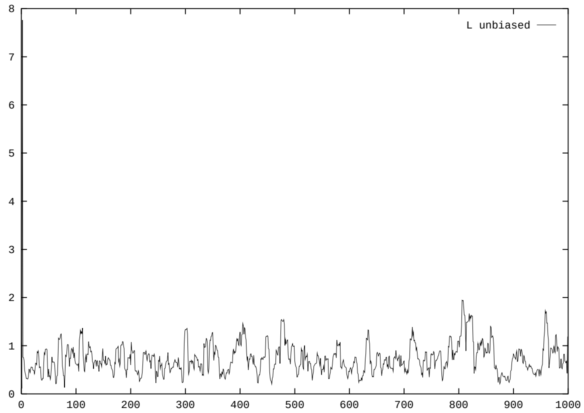

Figure 5. Vector magnitude sequence filtered using unbiased L-filter.