BERNOULLI SHIFT GENERATED CHAOTIC WATERMARKS: THEORETIC INVESTIGATION

S. Tsekeridou, V. Solachidis, N. Nikolaidis, A. Nikolaidis, A. Tefas and I. Pitas

Department of Informatics
Aristotle University of Thessaloniki
Box 451, 54006 Thessaloniki, Greece
e-mail: pitas@zeus.csd.auth.gr

ABSTRACT

The paper statistically analyzes the behaviour of chaotic watermark signals generated by n-way Bernoulli shift maps. For this purpose, a simple blind copyright protection watermarking system is considered. The analysis involves theoretical evaluation of the system detection reliability, when a correlator detector is used. The aim of the paper is twofold: (i) to introduce the n-way Bernoulli shift generated chaotic watermarks and theoretically contemplate their properties with respect to detection reliability and (ii) to theoretically establish their potential superiority against the widely used pseudorandom watermarks. Experimental verification of the theoretical analysis results is also performed.

1. INTRODUCTION

The risk of illegal copying, reproduction and distribution of copyrighted material is becoming more threatening with the all-digital evolving solutions adopted by content providers, system designers and users, thus creating a pressing demand for copyright protection of multimedia content. This demand has been lately addressed by the emergence of a variety of watermarking methods. The main trend is to use pseudorandom watermarks, which attain important properties for a watermarking application, such as autocorrelation function in the form of a Dirac delta function, unpredictability and statistical undetectability. Furthermore, the most widely used detector is the correlator. Up to now, the foundation for performance evaluation of the majority of watermarking methods has been mainly experimental without theoretical justification of their efficiency. Only few ones have attempted to statistically analyze the performance of image watermarking schemes [1, 2].

This paper statistically analyzes the behaviour of chaotic watermark signals generated by n-way Bernoulli shift maps. For this purpose, a simple blind copyright protection watermarking system is considered. The analysis involves theoretical evaluation of the system detection reliability, when a correlator detector is used. The effect of distortions (lowpass filtering, noise corruption) on the detection reliability is also theoretically investigated. The aim of the paper is twofold: (i) to introduce the n-way Bernoulli shift generated chaotic watermarks and theoretically contemplate their properties with respect to detection reliability and (ii) to theoretically establish their potential superiority against the widely used pseudo-random watermarks. Chaotic watermarks attain similar desirable properties with pseudorandom ones with the additional feature of controllable spectral/correlation properties.

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2. WATERMARKING SYSTEM MODEL

Let X_i , $i \in [1, N]$ be samples of the 1-D continuous valued host signal $\mathbf{X}$ of length N. Let also V_i , $i \in [1, N]$ be the samples of the continuous valued watermark signal $\mathbf{V}$ generated by a watermark generation function g: $\mathbf{V} = g(N, K)$. K denotes the watermark key. The watermark signal is assumed to be zero mean i.e. $E[V_i] = 0$. We assume that $\mathbf{V}$ is embedded additively to $\mathbf{X}$, thus generating the watermarked signal $\mathbf{X}_w = \mathbf{X} + p\mathbf{V}$, where p is a factor that determines the watermark strength.

Given a signal $\mathbf{S}'$, watermark detection aims at finding whether $\mathbf{S}'$ hosts a certain watermark $\mathbf{W}' = g(N, K')$. Thus, watermark detection can be formulated as a hypothesis test, the two hypotheses (events) being the following:

- H_0: $\mathbf{W}'$ is indeed embedded in $\mathbf{S}'$.
- H_1: $\mathbf{S}'$ does not host $\mathbf{W}'$.

Event H_1 occurs either if $\mathbf{S}'$ is not watermarked (event H_{1a}) or if $\mathbf{S}'$ is marked with a different watermark $\mathbf{W} = g(N, K'')$ than the one that we are trying to detect (event H_{1b}). Thus $H_1 = H_{1b} \cup H_{1a}$ where H_{1a}, H_{1b} are mutually exclusive. If the host signal has not been distorted, the form of $\mathbf{S}'$ for the three events H_0, H_{1a}, H_{1b} can be summarized in: $\mathbf{S}' = \mathbf{S} + p\mathbf{W}$. For $p = 0$, this results in $\mathbf{S}' = \mathbf{S}$ which corresponds to H_{1a}, whereas for $p \neq 0$ and $\mathbf{W} = \mathbf{W}'$, $\mathbf{S}' = \mathbf{S} + p\mathbf{W}'$ which corresponds to H_0. Finally, for $p \neq 0$ and $\mathbf{W} \neq \mathbf{W}'$, we get the form of $\mathbf{S}'$ under H_{1b}.

During watermark detection, the correlation c between $\mathbf{S}'$ and $\mathbf{W}'$ is calculated: $c = \frac{1}{N}\sum_{i=1}^{N}(S_i W_i' + pW_i W_i')$. In order to decide on the valid hypothesis, c is compared against a suitably selected threshold T. The system performance for a given threshold can be measured in terms of the false alarm probability $P_{fa}(T)$, (i.e., the probability to detect a watermark in a signal that is not watermarked or is watermarked with a different watermark) and the probability of false rejection $P_{fr}(T)$ (i.e., the probability to erroneously neglect the watermark existence in the signal):

$$P_{fa}(T) = \text{Prob}\{c > T | H_1\} = \int_T^{\infty} f_{H_1}(t)dt \quad (1)$$

$$P_{fr}(T) = \text{Prob}\{c < T | H_0\} = \int_{-\infty}^{T} f_{H_0}(t)dt \quad (2)$$

f_{H_0}, f_{H_1} are the probability density functions of c under hypotheses H_0, H_1 respectively. By solving (1), (2) for the independent variable T and equating the results, P_{fr} can be expressed as a function of P_{fa}. The plot of P_{fa} versus P_{fr} is called the receiver operating characteristic (ROC) curve of the watermarking system. This curve conveys all the necessary system performance information. For the studied watermark sequences, f_{H_0}, $f_{H_{1a}}$, $f_{H_{1b}}$ are normal distributions, as will be shown. Then, the ROC curve is given by:

$$P_{fa} = \frac{1}{2}[1 - \mathrm{erf}[\frac{\sqrt{2}\sigma_{H_0}\mathrm{erf}^{-1}(2P_{fr} - 1) + \mu_{H_0} - \mu_{H_1}}{\sqrt{2}\sigma_{H_1}}]] \quad (3)$$

The mean value of c, μ_c, and its variance, σ_c^2, for both H_0 and H_1, assuming independence between $\mathbf{W}$ and $\mathbf{S}$, are given by:

$$\mu_c = \frac{1}{N}\sum_{i=1}^{N} E[S_i]E[W'_i] + \frac{1}{N}\sum_{i=1}^{N} pE[W_i W'_i] \quad (4)$$

$$\begin{aligned} \sigma_c^2 = & \frac{1}{N^2}[\sum_{i=1}^{N}(E[S_i^2]E[{W'}_i^2] + p^2 E[{W'}_i^2 W_i^2] + \\ & 2pE[S_i]E[W_i {W'}_i^2]) + \\ & \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N}(E[S_i S_j]E[W'_i W'_j] + \\ & pE[S_i]E[W'_i W_j W'_j] + pE[S_j]E[W_i W'_j W'_i] + \\ & p^2 E[W_i W_j W'_i W'_j])] - \mu_c^2 \end{aligned} \quad (5)$$

An assumption about the statistical properties of $\mathbf{S}$ has also to be made: it is assumed wide-sense stationary, i.e., $\mu_S = E[S_i]$, $E[S_i S_{i+k}] = \Gamma_{S,S}(k), \forall i \quad i \in [1, N]$, described by a first order separable autocorrelation function [1]:

$$\Gamma_{S,S}(k) = \mu_S^2 + \sigma_S^2 \alpha^k, k \geq 0 \quad (6)$$

where σ_S^2 is the signal variance and $\alpha = 0.9, \cdots, 0.99$.

3. WATERMARKS GENERATED BY N-WAY BERNOULLI SHIFT MAPS

n-way Bernoulli shifts $B_n(r)$ are chaotic maps defined by: $\mathcal{B}_n : [0,1] \to [0,1]$, $r' = B_n(r) = nr(\mathrm{mod}\ 1)$. This map belongs to the class of piecewise affine Markov maps. A watermark sequence is generated by the map's recursive application:

$$W_{i+1} = B_n(W_i) = nW_i(\mathrm{mod}\ 1) \quad i \in [1, N] \quad (7)$$

The sequence starting point W_1 (map's initial condition) is considered as the watermark key K. The uniform distribution is an invariant probability density for the n-way Bernoulli shift maps [3, 4]. Watermark signals (Bernoulli chaotic watermarks), generated in this way, are wide-sense stationary. To attain zero mean, the following modification is done:

$$\begin{aligned} & \mathcal{B}_n : [-0.5, 0.5] \to [-0.5, 0.5] \\ & W_{i+1} = B_n(W_i) = n(W_i + \tfrac{1}{2})(\mathrm{mod}\ 1) - \tfrac{1}{2} \end{aligned} \quad (8)$$

Sample W_{i+k} is derived from sample W_i through [4]:

$$W_{i+k} = n^k(W_i + 0.5)(\mathrm{mod}\ 1) - 0.5 \quad k > 0 \quad (9)$$

Thus, the output of an n-way Bernoulli shift map after k iterations (denoted by B_n^k) is equal to that of a n^k-way Bernoulli shift map:

$$W_{i+k} = B_n^k(W_i) = B_{n^k}(W_i) \quad k > 0 \quad (10)$$

If the starting point W_1 of the map is an irrational number, the generated sequence exhibits a chaotic, non-periodic behaviour [5]. Thus, if one considers two Bernoulli chaotic watermarks $\mathbf{W}$, $\mathbf{W}'$ generated by the iterative application of the same map B_n on two distinct, irrational starting points (watermark keys) W_1, W'_1, both belonging to the same chaotic orbit, there will always be an integer $k > 0$ such that:

$$W'_1 = B_n^k(W_1) \quad \text{OR} \quad W_1 = B_n^k(W'_1) \quad (11)$$

Consequently, their samples W'_i, W_i will also be associated by ($\forall i, i \in [1, N]$):

$$W'_i = W_{i+k} = B_n^k(W_i) \quad \text{OR} \quad W_i = W'_{i+k} = B_n^k(W'_i) \quad (12)$$

These corollaries are used for the derivation of joint moments of Bernoulli chaotic watermarks. Bernoulli maps are characterized by uniform invariant density. Based on this, the m-order moments for W_i are calculated:

$$E[W_i^m] = \int_{-0.5}^{0.5} x^m f(x)dx = \begin{cases} 0 & m \ \text{odd} \\ \frac{1}{(m+1)2^m} & m \ \text{even} \end{cases} \quad (13)$$

The derivation of the joint moments of $\mathbf{W}$, $\mathbf{W}'$ appearing in (4), (5), for Bernoulli chaotic watermarks is reported in [6]. One of the joint moments is the autocorrelation function $R_{WW}(k)$:

$$R_{WW}(k) = E[W_i W_{i+k}] = \frac{1}{12n^k} \quad k \geq 0 \quad (14)$$

By observing (14), one concludes that W_i and W_{i+k} are correlated for small values of k and n. Convergence, though, occurs quickly as k increases, even for small n. As n increases, the autocorrelation function of the Bernoulli chaotic watermarks approximates a Dirac delta function, i.e., the autocorrelation function, ideally, of random watermarks. Thus, n controls the correlation properties of Bernoulli chaotic watermarks. Based on that, it is easily derived that, for small values of n, Bernoulli chaotic watermarks are characterized by lowpass spectrums, while, as n increases, the latter tend to be white, thus converging towards the spectrum of random watermarks. In short, it is concluded that n controls the correlation/spectral properties of Bernoulli chaotic watermarks. By appropriately choosing n, Bernoulli chaotic watermarks can attain the best possible performance for the application at hand.

Substituting the evaluated expressions in (4) and (5), analytical expressions for μ_c and σ_c^2 are derived for a watermarking system based on Bernoulli chaotic watermarks:

$$\mu_c = \frac{p}{12n^k} \quad (15)$$

Due to its extensive length, the expression of σ_c^2 is ommitted. The reader may consult [6]. These expressions can be used to obtain μ_c and σ_c^2 for events H_0 ($\mathbf{W} = \mathbf{W}'$) by setting $k = 0$, H_{1b} ($\mathbf{W} \neq \mathbf{W}'$) by setting $k \neq 0$ and H_{1a} by setting $p = 0$. σ_c^2 for event H_{1b} proves to be greater than that for event H_{1a}. μ_c is larger for small $k > 0$ but converges to that for H_{1a}, as k increases. Thus, event H_{1b} is the worst case in terms of bigger probability errors than H_{1a} or H_1.

Although Bernoulli chaotic watermarks prove to be correlated for small $k > 0$, the Central Limit Theorem for random variables with small dependency [7] may be used in order to establish that c attains a Gaussian distribution, even for event H_{1b} (assuming that N is sufficiently large). Furthermore, under the worst case assumption, both μ_c and σ_c^2 converge to a constant value for large k. For such k, $R_{WW}(k) = 0$ meaning that $E[W_i W_j] = E[W_i]E[W_j] = 0$. Thus, the terms of the sum in c can be considered sufficiently independent and the distribution of c under event H_{1b} for $k \to \infty$ can be assumed normal. In such a case, $P_{fa,H_{1b}}$ can be estimated using the limit values of μ_c and σ_c^2 as $k \to \infty$. This is done since convergence is actually quickly reached leading to a very small probability (for large N) of actually facing a case where k is rather small. P_{fr} values are estimated using the values of μ_c and σ_c^2 for $k = 0$ and ROC curves are evaluated from (3).

4. PSEUDORANDOM WATERMARKS

Zero-mean pseudorandom sequences in $[-0.5, 0.5]$ are considered. Such sequences attain a white spectrum. Furthermore, for such watermarks, the terms of the sum in c can be safely assumed to be sufficiently independent. Thus, due to the Central Limit Theorem, c attains a Gaussian distribution for a sufficiently large N. Based on their properties, the moments of (4), (5), are easily obtained:

$$E[W_i^m] = \begin{cases} 0 & m \text{ odd} \\ \frac{1}{(m+1)2^m} & m \text{ even} \end{cases} \quad (16)$$

$$E[W_i^l W_j^m] = E[W_i^l]E[W_j^m] \quad (17)$$

Similar expressions can be derived for the moments involving more than two random variables. Using these, μ_c and σ_c^2 for a watermarking system based on random watermarks can be calculated:

$$\mu_c = \begin{cases} \frac{p}{12} & \text{if } \mathbf{W} = \mathbf{W}' \\ 0 & \text{if } \mathbf{W} \neq \mathbf{W}' \\ 0 & \text{if } p = 0 \end{cases} \quad (18)$$

$$\sigma_c^2 = \begin{cases} \frac{1}{12N}(\mu_S^2 + \sigma_S^2 + \frac{p^2}{15}) & \text{if } \mathbf{W} = \mathbf{W}' \\ \frac{1}{12N}(\mu_S^2 + \sigma_S^2 + \frac{p^2}{12}) & \text{if } \mathbf{W} \neq \mathbf{W}' \\ \frac{1}{12N}(\mu_S^2 + \sigma_S^2) & \text{if } p = 0 \end{cases} \quad (19)$$

It is seen that c attains the same mean value for both events $\mathbf{W} \neq \mathbf{W}'$ and $p = 0$, while its variance for the first event is larger than that for the second, proving that the first event is the worst case.

5. NOISE ADDITION

$\mathbf{S}' = \mathbf{S} + p\mathbf{W}$ is assumed to be corrupted by additive random white i.i.d. noise ϵ uniformly distributed in the interval $[-\epsilon_r, \epsilon_r]$. ϵ has zero mean value, $\mu_\epsilon = 0$, and variance equal to $\sigma_\epsilon^2 = \epsilon_r^2/3$. $\mathbf{W}$ may be either a chaotic watermark or a random one. Detection involves estimation of the correlation between the noise corrupted signal $\mathbf{S}' + \epsilon$ and a watermark $\mathbf{W}'$, i.e.: $c_\epsilon = \frac{1}{N}\sum_{i=1}^N (S_i' + \epsilon_i)W_i'$. ϵ does not modify the correlator's Gaussian distribution since it is independent of the other signals. In order to determine the influence of noise addition on the system's detection reliability, the mean value μ_{c_ϵ} and variance $\sigma_{c_\epsilon}^2$ of c_ϵ must be estimated:

$$\mu_{c_\epsilon} = \frac{1}{N}\sum_{i=1}^N \left[E[S_i]E[W_i'] + E[\epsilon_i]E[W_i'] + pE[W_i W_i']\right] \quad (20)$$

For Bernoulli chaotic watermarks, (20) leads to $\mu_{c_\epsilon} = \frac{p}{12n^k}$. For random watermarks, we obtain:

$$\mu_{c_\epsilon} = \begin{cases} \frac{p}{12}, & \text{if } \mathbf{W} = \mathbf{W}' \\ 0, & \text{if } \mathbf{W} \neq \mathbf{W}' \text{ or } p = 0 \end{cases} \quad (21)$$

Its variance $\sigma_{c_\epsilon}^2$ is given by:

$$\begin{aligned} \sigma_{c_\epsilon}^2 = & \frac{1}{N^2}E[\sum_{i=1}^N ((S_i + pW_i + \epsilon_i)W_i')^2 + \\ & \sum_{i=1}^N \sum_{j=1, j\neq i}^N ((S_i + pW_i + \epsilon_i)W_i') \\ & ((S_j + pW_j + \epsilon_j)W_j') - \mu_{c_\epsilon}^2 = \sigma_c^2 + \frac{\epsilon_r^2}{36} \end{aligned} \quad (22)$$

where σ_c^2 denotes the correlator variance under no distortions. It is seen that noise corruption is not a serious threat, since it does not affect the mean value of the correlation and only slightly influences its variance, provided that its power is much less than that of the original signal. Furthermore, noise corruption affects the system performance similarly for either random or chaotic watermarks.

6. LINEAR LOWPASS FILTERING

We consider a moving average filter of length $2F + 1$ with the impulse response $h_i = \frac{1}{2F+1}$, $i \in [-F, F]$. The filtered signal $\mathbf{S}'_f$ is obtained by the linear convolution: $S'_{f,i} = S_i' * h_i = \sum_{l=-F}^F h_l S'_{i-l}$, $i \in [1, N]$. The correlator detector estimates now the correlation c_f between $\mathbf{S}'_f$ and a watermark $\mathbf{W}'$: $c_f = \frac{1}{N}\sum_{i=1}^N \sum_{l=-F}^F h_l(S_{i-l} + pW_{i-l})W_i'$ Filtering does not also modify the correlator Gaussian distribution. μ_{c_f} and $\sigma_{c_f}^2$ are now evaluated by:

$$\mu_{c_f} = \frac{1}{N}\sum_{l=-F}^F h_l \sum_{i=1}^N (E[S_{i-l}]E[W_i'] + pE[W_{i-l}W_i']) \quad (23)$$

For the case of Bernoulli chaotic watermarks, (23) becomes:

$$\mu_{c_f} = \begin{cases} \frac{p}{12(2F+1)}\frac{1}{n^k}\left[1 + \frac{1-(\frac{1}{n})^F - n + n^{F+1}}{n-1}\right], & k \geq F \\ \frac{p}{12(2F+1)}\frac{1}{n^k}\left[1 + \frac{1-(\frac{1}{n})^F - n + n^{k+1} + n^{2k}(1-(\frac{1}{n})^{F-k})}{n-1}\right], & 0 \leq k < F \end{cases} \quad (24)$$

When random watermarks are used, (23) leads to:

$$\mu_{c_f} = \begin{cases} \frac{p}{12(2F+1)}, & \text{if } \mathbf{W} = \mathbf{W}' \\ 0, & \text{if } p = 0 \text{ or } \mathbf{W} \neq \mathbf{W}' \end{cases} \quad (25)$$

Estimation of $\sigma_{c_f}^2$ is performed as follows:

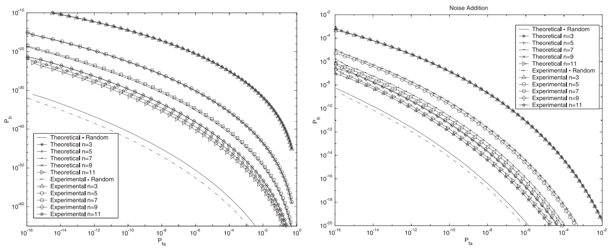
$$\begin{aligned} \sigma_{c_f}^2 = & \sum_{l=-F}^F h_l^2 E[c^2] + \frac{1}{N^2}\sum_{m=-F}^F \sum_{n=-F, n\neq m}^F h_m h_n \\ & \Big[\sum_{i=1}^N (E[S_{i-m}S_{i-n}]E[W_i'^2] + p^2 E[W_{i-m}W_{i-n}W_i'^2]) + \\ & \sum_{i=1}^N \sum_{j=1, j\neq i}^N (E[S_{i-m}S_{j-n}]E[W_i'W_j'] + \\ & p^2 E[W_i'W_j'W_{i-m}W_{j-n}])\Big] - \mu_{c_f}^2 \end{aligned} \quad (26)$$

In (26), $E[c^2] = \sigma_c^2 + \mu_c^2$ denotes the 2nd moment of c under no distortions for any kind of watermarks. For the case of Bernoulli chaotic watermarks, $\sigma_{c_f}^2$ has been numerically estimated for predefined values of N, n and F. Such numerical estimation is prone to errors dependent on the computational accuracy. For random watermarks, (26) leads to an expression outlined in [6], due to paper length constraints.

7. EXPERIMENTAL RESULTS

In order to empirically verify the watermarking system performance in terms of ROC curves, the system is fed with two input signals: a music audio signal and a random uniformly-distributed signal, appropriately prefiltered to comply with model (6). Slightly better performance is obtained in the case of perfect validity of (6), i.e. random signal. The system performance is measured for both Bernoulli and random watermarks to enable their comparison. The value of p is set such that the watermarked signal has SNR=30dB. 10000 keys are used in the experiments. ROC curve estimation is performed under the worst case assumption, (event H_{1b}).

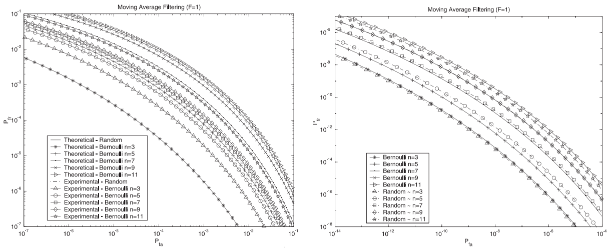
The achieved ROC curves under no attack, for varying n and the audio signal as input, are shown in Figure 1a. The coincidence of the theoretical and empirical ones is noted. Respective curves are illustrated for random watermarks. It is seen that the latter attain the best performance with respect to the overall error probability. However, as n increases, the performance of Bernoulli watermarks is quickly converging to that of random ones, since their lowpass spectral properties tend to white with increasing n. In Figure 1b, respective results are shown after noise addition. Similar conclusions are drawn. Noise of small power does not greatly affect the system performance: P_{fa} and P_{fr} values are only slightly increased.



(a) (b)

Fig. 1. Theoretical and empirical ROCs for Bernoulli chaotic and random watermarks after (a) no attack, (b) noise addition.

The system performance is also evaluated after lowpass filtering. Figure 2a shows both the theoretically and experimentally derived ROC curves after moving average filtering (length 3) of the watermarked audio signal for Bernoulli and random watermarks. Both types of results tend to be similar. The numerical estimation of the theoretical $\sigma_{c_f}^2$ for Bernoulli watermarks has its impact on the ROC curves. Use of the latter leads to better performance, especially for small values of n, compared to random watermarks. This is easily justified by their lowpass spectral properties for small n, which renders them robust against lowpass distortions. In order to establish this, lowpass filtering of random watermarks prior to embedding was done, so that they attain similar spectral/correlation properties with Bernoulli watermarks for various n. Figure 2b shows the experimentally estimated ROC curves for LP random watermarks and for varying n, after moving average filtering of length 3 and for the random signal as input, compared against the ones obtained when using respective Bernoulli watermarks. They are nearly identical proving that the watermark spectrum plays a significant role in watermarking applications.



(a) (b)

Fig. 2. (a) Theoretical and empirical ROC curves for Bernoulli chaotic and random watermarks after lowpass filtering, and (b) empirical ROC curves for Bernoulli chaotic (solid lines) and random watermarks (dashed lines), prefiltered to attain similar spectral/correlation properties with the chaotic ones.

8. CONCLUSIONS

The n-way Bernoulli shift generated chaotic watermarks are introduced and their statistical properties relevant to a watermarking application are investigated to determine the overall system performance under no distortions or simple attacks. Similar investigation is performed for pseudorandom watermarks, for comparison purposes. Bernoulli chaotic watermarks attain controllable spectral/correlation properties dependent on the value of n. They can be easily constructed with the appropriate spectral properties according to the prospective application or potential distortion, while preserving their invariant probability density. Random watermarks should be preprocessed to exhibit similar efficiency. Such preprocessing modifies their initial probability distribution which enables the risk of possible perceptibility.

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