MORPHOLOGICAL VORONOI TESSELLATION AND DELAUNAY TRIANGULATION

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ABSTRACT

A new method is introduced for implementing the Voronoi tessellation and the Delaunay triangulation. in the Z^2 plane. This method uses metrics on a discrete grid and discrete approximations of the euclidean metric. Mathematical morphology is used to implement Voronoi tesseliation based on these metrics.

1. INTRODUCTION

Voronoi tessellation is a very important tool in computational geometry [4], object recognition [1] and image analysis [2]. Several important problems can be solved by employing Voronoi tessellation, for example Delaunay triangulation, convex hull, object decomposition into simple components (triangles).

Let $X=\{x_1, x_2, \ldots, x_N\}$ be a set of N points on a subset W. The **Voronoi Tessellation** is given by:

$$V(i)=\{x \in W: d(x,x_i)<d(x,x_j),\ j \neq i\} \tag{1}$$

$$Vor(X)=\cup V(i)$$

where $W \subseteq R^n$ or $W \subseteq Z^n$ and d() is a distance function. V(i) is the Voronoi region of x_i and Vor(X) is called the Voronoi diagram of X.

In this paper a novel algorithm for the computation of Voronoi tessellation is proposed. It is based on mathematical morphology. It construct Voronoi diagrams on the Euclidean grid Z^2 for any distance measure, e.g. Euclidean, Hausdorff, cityblock, chess-board, octagonal. Its computational complexity is of order $O(1/N)$.

Each Voronoi region V(i) contains all points of W that are closer to x_i than to any other x_j, $j \neq i$. This means that it can be obtained by "growing" all points x_i, $i=1, \ldots, N$ simultaneously until they occupy the entire W. When two Voronoi region collide, a boundary is formed and no further growth is allowed along this boundary. The growth mechanism is the **dilation** operator:

$$Y \oplus B^s = \bigcup_{b \in B} Y_{-b} = \left\{ x \in Z^2 : B_x \cap Y \neq \emptyset \right\} = \left\{ x : B_x \uparrow Y \right\} \tag{2}$$

In the following section a presentation of distance functions is

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made. In section 3 a method for implementing the Voronoi tessellation by using various distance functions is given. Simulation examples are shown and conclusions are drawn in section 4.

2. DISTANCE FUNCTIONS

The notion of the distance between two points is fundamental in a number of geometrical problems. In the case of the Voronoi diagram, the Euclidean metric is used to find the points lying on the perpendicular bisector of two given points in R^2.

The best known distance measure between two points in R^2 is the Euclidean distance. A coordinate independent distance that can be used to calculated the Euclidean distance is the Hausdorff distance function, that is defined by using mathematical morphology

$$d_h(x,y)=\inf\left\{\rho \ : \ x\oplus B(\rho)\uparrow y\right\} \tag{3}$$

where $B(\rho)$ is a disk of radius $\rho\in R^2$.

In order to define a distance measure in Z^2, grids with different connectivity are used. Such are the city block for a 4-connected and the chessboard distance for an eight-connected grid. When trying to approximate in Z^2 the connectivity of the R^2 grid, then the octagonal distance function has to be used. The best approximation of the Euclidean measure in Z^2 is given by:

$$d_z(x,y)=\left\{n\in\mathbb{N}: \ n-0.5\leqslant d_e(x,y) < n+0.5\right\} \tag{4}$$

where d_e is the Euclidean distance of x,y in Z^2. An equivalent definition for (3) in Z^2 is:

$$d_h(x,y)=\inf\left\{n \ : \ x\oplus B(n)\uparrow y\right\} \quad x,y\in Z^2,\ n=0,1,2,\ldots \tag{5}$$

where $B(n)$ defines a structuring element of size n. There are several ways to construct a structuring element $B(n)$ of size n. In the following we present two distance functions based on (5) but using a different definition for $B(n)$.

The uniform-step distance (USD) is denoted by $d_{usd}(x,y)$ and defined by using (5), as follows [5]:

$$B(n)=\begin{cases} B\oplus B\oplus\ldots\oplus B \ \ (n \text{ times}), & n=1,2,\ldots \\ 0, & n=0 \end{cases} \tag{6}$$

where B is the structuring function of unit size. B is a symmetric compact set in Z^2 containing the origin. When using a RHOMBUS (SQUARE) structuring element [3] then the city block (chessboard) distance function results. Eqs. (5)-(6) provide an easy iterative way for computing different measures by using dilation.

Another distance function is the periodically-uniform-step distance (PUSD). If $B_1, B_2, \dots, B_m$ are all symmetric compact sets in Z^2 containing the origin, B(m) of size m ($m \geq 1$) is given by [5]:

$$B(m) = B_1 \oplus B_2 \oplus \dots \oplus B_m, \qquad (m \geq 1), \qquad B_1 \subset B_2 \subset \dots \subset B_m \tag{7}$$

For $n \geq m$ B(n) is defined by:

$$B(n) = \begin{cases} pB(m) \oplus B(q), & n = pm + q > 0,\ q < m, \\ 0, & n = 0 \end{cases} \tag{8}$$

where $pB(m) = B(m) \oplus B(m) \dots \oplus B(m)$, (p-times). If B_1, B_2 (m=2) are the RHOMBUS and the SQUARE structuring elements [3], then the octagonal distance function is evaluated. Eqs. (5),(7) and (8) can approximate the Euclidean distance in the grid Z^2 if the sets $B_1, \dots, B_m$ are chosen carefully. A better approximation of the Euclidean distance in Z^2 can be found by using another class of distance functions based on a slightly different definition:

$$d_a(x,y) = \inf \left\{ k : X_k \uparrow y \right\}, \qquad k \in N$$
$$X_k = (X_{k-1} \oplus B_k) \cup S_k^+ \qquad k > 0, \qquad X_0 = x \tag{9}$$

where B_k, k=1,2,... are symmetric structuring elements which contain the origin and S_k^+ is a set of points to be defined. When each structuring element B_k is chosen to be a RHOMBUS and denoted by B the set S_k^+ is given by:

$$S_k^+ = \left\{ z \in Z^2 : z \in ((X_{k-1} \oplus 2B) - (X_{k-1} \oplus B)) \text{ and } d_z(z,x) = k \right\} \tag{10}$$

where $2B = B \oplus B$ and $d_z()$ is given by (4). If S_k^+ is chosen as in (10), then X_k in (9) implements a recursive way for growing a disk in Z^2, centered on the point x. At each step k, the disk X_{k-1} is dilated by B and the points of S_k^+ given by (10) are appended. The sets S_k^+, k=1,2,... can be precomputed and stored.

In the following section the above metrics will be applied to the Voronoi tessellation.

3. VORONOI TESSELLATION

A new method for constructing the Voronoi diagram of a given set of distinct points in $X \subset W \subseteq Z^2$ will be presented. This method finds the Voronoi regions of a given set $X \subset W$ rather than the Voronoi edges and vertices. It uses successive region growing of the n-Voronoi region of each point, denoted as $N_n(i)$. The $N_n(i)$ region of a point x_i is the set of points already appended to x_i during the n previous growing steps. When two or more n-Voronoi regions collide, the collision points form subsets of the Voronoi polygons and the growing stops in this direction. This procedure is repeated until no further growth is possible in $W \subseteq Z^2$. The overall algorithm is described as follows:

$$n_n(i) = \left\{ x \in (W-X):\ x \in [n_{n-1}(i) \oplus B_n],\ x \notin e_n(i),\ x \notin \underset{x_j \in (X-x_i)}{N_{n-1}(i)} \right\}, \quad n \geq 1$$

$$n_0(i) = x_i \tag{11}$$

$$N_n(i) = N_{n-1}(i) \cup n_n(i), \qquad N_0(i) = \{\ \}$$

where B_n is the n-th in order structuring element of B(n). The set e_n contains the collision points of x_i at the growing step n:

$$e_n(i) = \left\{ x \in (W-X):\ x \in \bigcup_{\substack{x_j \\ x_j \in (X-x_i)}} \left[D_n(i) \cap D_n(j) \right] \right\} \tag{12}$$

$$D_n(i) = \{ x \in W:\ x \in X_n - X_{n-1} \} \tag{13}$$

$$X_n = x_i \oplus B(n)$$

The points in the set $e_n(i)$ have the same distance n from a point x_i and at least from another point x_j of X ($i \neq j$) by using (5). The set $N_n(i)$ contains all the points of (W-X) that have been appended to x_i and are at a distance $k \leq n$. If $k_{max}(i)$ denotes the step in which the point x_i cannot grow any more, the Voronoi region V(i) of a point $x_i \in X$ is defined by using (11) as:

$$V(i) = N_{k_{max}}(i) \triangleq N_{k_{max}(i)}(i) \tag{14}$$

Thus the union of all $k_{max}(i)$-Voronoi regions of the points in X is equal to the Voronoi diagram of the set X. (11) is useful in practice because it allows the construction of the Voronoi neighborhoods recursively. A n-Voronoi neighborhood $n_n(i)$ contains the new points to be appended to the corresponding (n-1)-Voronoi region $N_{n-1}(i)$. Thus the $N_{n-1}(i)$ region grows to the $N_n(i)$ region. The set which contains the **boundary points** of V(i) is denoted by F(i) and is given by:

$$F(i) = \left\{ x \in (W-X) : x \in (N_{k_{max}}(i) - N_{[k_{max}-1]}(i)) \right\} \tag{15}$$

where (-) denotes set subtraction.

Let $X = \{x_1, x_2, \ldots, x_N\}$ be a set of N points. The basic idea of the algorithm is to grow each given point of X by growing the set $N_n(i)$. More precisely, at each step the points of $n_n(i), \forall\, x_i \in X$ are appended to $N_{n-1}(i)$, $\forall\, x_i \in X$, thus resulting in the $N_n(i)$ regions, $\forall\, x_i \in X$. The set $n_n(i)$ is found by dilating $n_{n-1}(i)$ by B_n and then checking whether all points $x \in [n_{n-1}(i) \oplus B_n]$ have been appended to x_i

exclusively (and not to another point x_j, $j \neq i$ at the same time). We also check if the point x already belongs to a (n-1)-Voronoi region of a point x_j, $j \neq i$ of X. The iterative dilations stop when all points of W-X have been appended to one of the Voronoi regions.

The previously described method can be modified to construct the Euclidean Voronoi diagram in Z^2. In this case the distance function used to derive the definitions is given by (9). The only change to the above formulas is the set X_k, now defined:

$$\text{for } n=1: X_1=(x_i \oplus B) \cup S_1^+ \qquad (16)$$
$$\text{for } n>1: X_n=(X_{n-1} \oplus B) \cup S_k^+$$

where S_k^+ is given by (10) and B is the RHOMBUS structuring element.

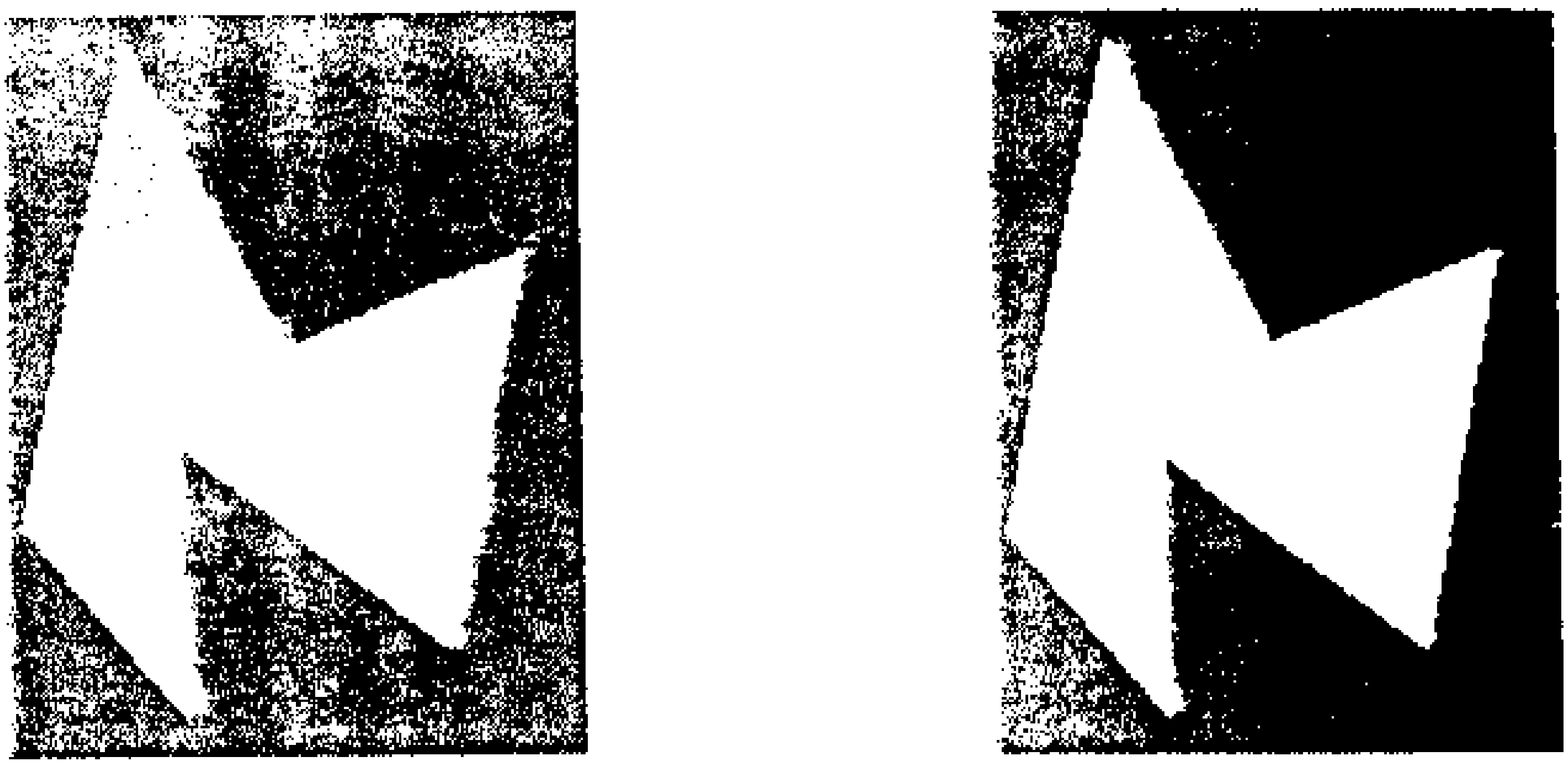

Figure 1: (a) Voronoi tessellation (b) Delaunay triangulation of binary object.

4. SIMULATIONS EXAMPLES AND CONCLUSIONS

The previously described method has been implemented in C programming language. The Morphological Voronoi tessellation using this method has been tested and found to be very successful. Figure 1a. This tessellation has also been used to obtain the Delaunay triangulation [4]. The Delaunay triangulation of a polygonal object X obtained by using this method is shown in Figure 1b. The set X_c of the corners of this object have been obtained by morphological operations:

$$X_c = X_E \cup X_{E'},$$
$$X_E = X - X_B \qquad (17)$$
$$X_{E'} = X^c - (X^c)_B$$

where X^c denotes the complement of X with respect to W and X_B denotes set opening [3]. The corner set X_c is used to obtain the Voronoi tessellation of X and, subsequently, its Delaunay triangulation.

In this paper a new method for performing Voronoi tessellation on a set of points in Z^2 has been presented. A general definition of a distance function has been used that is implemented by using the morphological operator of dilation as a growing mechanism. The proposed method is independent of the coordinate system. It allows the implementation of different tessellations based on different distance functions, e.g. the Euclidean, the Hausdorff, the octagonal, the chessboard and the city block. It has low computational complexity of order O(1/N) and allows parallel implementation.

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