SORTING NETWORKS USING NONLINEAR L_P MEAN COMPARATORS

M. Pappas[1] *I. Pitas*[2]

[1]Dept. of Informatics, University of Thessaloniki, Box 451, GR-54006 Thessaloniki, GREECE
E-mail: mpap@zeus.csd.auth.gr
[2]Dept. of Informatics, University of Thessaloniki, Box 451, GR-54006 Thessaloniki, GREECE
E-mail: pitas@zeus.csd.auth.gr

ABSTRACT

In certain signal processing applications there is a need for fast hardware implementations of sorting algorithms and networks. So far, classical minimum/maximum comparators have been utilized in various sorting network topologies. However, these comparators can not attain high speeds in operation, due to limitations in digital technology. This paper introduces the L_p comparators, which are based on the theory of nonlinear mean filters. It will be shown that the disadvantage of introducing errors is counter-balanced by their faster performance, when compared to the performance of classical comparators. A novel L_p comparator-based sorting network will also be presented, for the fast calculation of the median of a data set. In this implementation, the number of steps required to produce the ordered output is not related to the number of inputs.

1. INTRODUCTION

In many areas of signal processing there is a strong need for fast sorting algorithms and structures (e.g. median filtering) [1]. A solution exists in the form of *sorting networks*, a special case of sorting algorithms, where the sequence of comparisons performed has homogeneous structure [2].

The basic functional unit of a sorting network is the comparator. The sorting network performance depends mostly on the performance of the type of comparator utilized. Comparators are difficult to be used in analog switched filters, where no digital circuitry is used. This paper presents a new type of comparator, the L_p comparator, that can be materialized using analog circuitry. Thus, it can be used for high speed analog or hybrid signal processing.

The L_p comparator is based on the theory of a subclass of nonlinear mean filters, the L_p mean filters [3]. L_p operators have already been proposed as alternatives to the maximum/minimum operators in grayscale mathematical morphology [4, 5].

In essence, sorting networks provide a topology that is independent of the comparator technology utilized. Thus, it is trivial enough to "replace" conventional comparators with L_p comparators, with no further changes to the network.

The layout of this paper is as follows: some important definitions and explanations of terms are given in Section 2. In Section 3 an estimation of the ordered output approximation error is presented. A median approximation sorting network, based on the use of L_p comparators is given in Section 4. Finally, Section 5 presents some overall conclusions.

2. DEFINITIONS

Before we proceed, some important definitions should be presented. A sorting network has N inputs $x_1, \ldots, x_N$ and N outputs $x_{(1)}, \ldots, x_{(N)}$, where $x_{(N)}$ denotes the largest element of the set $\{x_1, \ldots, x_N\}$ and $x_{(1)}$ denotes the smallest element of the same set. Figure 1(a) depicts an odd-even transposition sorting network of $N = 5$ inputs, while Figure 1(b) shows a classical comparator.

The $L_p\{\cdot\}$ operator is defined as the nonlinear L_p mean of a set of positive real numbers A [3, 4, 5]:

$$L_p\{A\} \stackrel{\text{def}}{=} \left(\frac{\sum_{x \in A} x^p}{M}\right)^{1/p} \quad (1)$$

where M is the number of elements in A. It has been proved that for $p \geq 0$ the following inequality holds [6]:

$$\min\{A\} \leq L_{-p}\{A\} \leq L_p\{A\} \leq \max\{A\} \quad (2)$$

Figure 1(c) shows a L_p comparator. When p is large, L_{-p} converges to the min operator and L_p to the max operator. From equation (1) it is evident that if the network of Figure 1(a) utilizes L_p comparators, then estimates $\hat{x}_{(i)}$ of the ordered input samples $x_{(i)}$, will be produced at the network outputs, where $i \in \{1, \ldots, N\}$. It is also obvious from (1) that a L_p comparator can be materialized in hardware with adders, multipliers and nonlinear amplifiers that raise to a power. Since analog implementations of these elements exist, the performance benefits should be obvious. However, for small or medium values of p, L_p comparators introduce errors, according to (1).

3. ERROR ESTIMATION

3.1. Error of a single comparator

Before we proceed to estimate the overall output error introduced by the use of L_p comparators in a sorting network, instead of the conventional ones, we shall first examine the approximation error introduced by a single L_p comparator. The inputs $x_1,\ x_2$ are assumed to be iid random variables in the set $(0, 255]$, having a uniform probability density function (pdf). In this case, it can be shown that the errors $e_{\max}(x_1, x_2)$, $e_{\min}(x_1, x_2)$, between $x_{(i)}$ and its L_p approximation $\hat{x}_{(i)}$, can be expressed in terms of the inputs $x_1,\ x_2$:

$$e_{\max}(x_1, x_2) = x_{(2)} - \hat{x}_{(2)} \simeq c|x_1 - x_2| \geq 0 \quad (3)$$

$$e_{\min}(x_1, x_2) = x_{(1)} - \hat{x}_{(1)} \simeq d|x_1 - x_2| \leq 0 \quad (4)$$

where the right-hand inequalities hold because of (2). Equations (3)-(4) state that the approximation errors can be expressed in terms of the absolute difference of the input samples. This is in par with the analysis that follows. From (1):

$$L_p\{x_1,\ x_2\} = \left(\frac{x_1^p + x_2^p}{2}\right)^{1/p} = \left(\frac{\max^p\{x_1, x_2\} + \min^p\{x_1, x_2\}}{2}\right)^{1/p} \quad (5)$$

Depending on the values of x_1, x_2 there are two cases. If the distance between x_1 and x_2 is large, that is $\max\{x_1, x_2\} \gg \min\{x_1, x_2\}$, then the following will hold:

$$\max^p\{x_1, x_2\} \gg \min^p\{x_1, x_2\} \quad (6)$$

Substitution of (6) into (5), yields:

$$L_p\{x_1,\ x_2\} \simeq \left(\frac{\max^p\{x_1, x_2\}}{2}\right)^{1/p} \quad (7)$$

Finally, by combining (3) and (7) the following expression can be deduced for the error $e_{\max}$:

$$e_{\max}(x_1, x_2) \simeq \left(1 - \frac{1}{2^{1/p}}\right) \max\{x_1, x_2\} \quad (8)$$

On the other hand, if $x_1 \simeq x_2$, then $\max\{x_1, x_2\} \simeq \min\{x_1, x_2\}$. This yields:

$$\max^p\{x_1, x_2\} \simeq \min^p\{x_1, x_2\} \quad (9)$$

and by (9) and (5):

$$L_p\{x_1,\ x_2\} \simeq \max\{x_1, x_2\} \quad (10)$$

Finally, combination of (3) and (10) yields:

$$e_{\max}(x_1, x_2) \simeq 0 \quad (11)$$

Equations (8) and (11) can be easily verified. In Figure 2 the value of $|e_{\max}|$ is plotted, as a function of the comparator inputs $x_1,\ x_2$, with $p = 5$. Indeed, the error is zero across the line $x_1 = x_2$ and increases almost linearly, as the distance from this line increases. It should be noted that we similar conclusions hold for the approximation error $e_{\min}$.

We must stress the fact that approximation errors depend largely on the value of the parameter p. Indeed, from (8) it is obvious that, as p increases, $e_{\max}$ converges to 0, regardless of the values of $x_1,\ x_2$.

3.2. Overall error

The real question regarding the use of L_p comparators in sorting networks, instead of the conventional (max/min) ones, is whether or not their error level is acceptable. From equations (3,4) it is obvious that errors introduced in the inputs of a L_p comparator accumulate with the errors introduced by the L_p element itself and propagate throughout the entire sorting network. As a direct result, the estimates of the largest (smallest) elements are produced at the network output with a negative (positive) bias. For example, if the largest input sample is $x_i = x_{(N)}$, then each time it passes a L_p comparator, a negative bias will be added to it. For large values of N and small values of p, it may be possible that the ordering of approximated output samples no longer holds. In this scenario, the network is rendered useless.

If an input sample x_j has a value close to the median, results are quite different. In this case, the number of comparisons with samples which have greater values is about the same with the number of comparisons with samples that have smaller values. If (3)-(4) are taken into account, then it can be stated that the number of times that the sample x_j will be negatively biased, will be the same as the number of times it will be positively biased. As a direct consequence, input samples with values close to the median, will appear at the network output with much smaller errors. Of course, bias is a function of both N and p.

3.3. Simulation results

In order to test the feasibility of replacing max/min with L_p comparators, about 5000 samples of uniform distribution, in the range 0.0001 to 255, were created by a pseudo-random number generator. The sorting network used was the odd-even transposition network of Figure 1(a). The *Mean Square Error* $MSE_{(i)}$ was utilized as a quantitative criterion of error at the ith output line of the network, that is:

$$\mathrm{MSE}_{(i)} = E[(x_{(i)} - \hat{x}_{(i)})^2] \simeq \frac{\sum_{k=1}^{K}(x_{(i)}^{(k)} - \hat{x}_{(i)}^{(k)})^2}{K}$$

where K is the number of test iterations. Since many non-linear image filtering schemes employ 3, 5 or 9-point filter windows (one dimensional and 3×3 two dimensional, respectively), tests were performed on sorting networks of N inputs, where N took values in the set $\{3, 5, 9\}$. In each case, parameter p took values in the set $\{2, 5, 8\}$. Errors increase with N and decrease with p. Acceptable error levels can be obtained when rather large values of p are used.

Results are quite different, if error compensation is introduced in the L_p comparator. The outputs of the L_p comparator take the following form:

$$\begin{aligned}\hat{x}_{(2)} &= L_p\{x_1, x_2\} + c|x_1 - x_2| \\ \hat{x}_{(1)} &= L_p\{x_1, x_2\} + d|x_1 - x_2|\end{aligned}$$

The constants $c,\ d$ can be found by minimizing the following functions, with respect to $c,\ d$:

$$E[(e_{\max}\{x_1, x_2\} - c\,|x_1 - x_2|)^2]$$
$$E[(e_{\min}\{x_1, x_2\} - d\,|x_1 - x_2|)^2]$$

If we assume that $x_1,\ x_2$ are iid random variables, where their pdf is known, then parameters $c,\ d$ can be estimated by numerical integration methods. The simulations of the previous paragraph were carried out, but this time modified L_p comparators were utilized. Results, in the case of $p \in \{5, 9\}$, are depicted in Figures 3(a)-(b). These results indicate that the approximation error is within acceptable levels. However, this improved accuracy comes at the cost of increased hardware complexity.

4. A MEDIAN APPROXIMATION NETWORK

L_p mean comparators can also be used to produce only the data median as follows. The operation of this median selection network is based on the following observation: if a set has N elements, then the total number of its subsets, consisting of ν elements each, equals $C(N, \nu) = \frac{N!}{\nu!(N-\nu)!}$, where $\nu = \frac{N+1}{2}$. In the ordered set representation, it is easy to verify that $x_{(\nu)}$ represents the median. We can show that there is at least one ν-element subset C_i of the N-element set, that satisfies:

$$\max\{C_i\} = x_{(\nu)},\ i \in \{1,\ldots,C(N,\nu)\} \tag{12}$$

while the rest $C(N,\nu)-1$ ν-element subsets satisfy:

$$\max\{C_j\} \geq \max\{C_i\},\ j \neq i,\ i,\ j \in \{1,\ldots,C(N,\nu)\} \tag{13}$$

If the $C(N,\nu)$ maxima $y_j = \max\{C_j\}$ are computed, then by equations (12)-(13):

$$x_{(\nu)} = \min\{y_j :\ 1 \leq j \leq C(N,\nu)\} \tag{14}$$

As an example, if $N = 3$ then $\nu = 2$ and $C(3,2) = 3$. Indeed, the two-element subsets of the three-element set $\{x_1, x_2, x_3\}$ are:

$$\begin{aligned} C_1 &= \{x_1, x_2\} \\ C_2 &= \{x_1, x_3\} \\ C_3 &= \{x_2, x_3\} \end{aligned}$$

and $x_{(2)} = \min\{\max\{C_1\}, \max\{C_2\}, \max\{C_3\}\}$.

4.1. Implementation cost

The L_p implementation requires $C(N,\nu)$ L_{p_1} comparators of ν inputs each and one L_{-p_2} comparator of $C(N,\nu)$ inputs. The input-output relationship is given by equation (1). Each L_{p_1} element operates in parallel with the rest $C(N,\nu)-1$ elements. Its ν inputs form a unique ν-element subset of the N-element set. The output of this element is an approximation of the subset maximum. Subsequently, the $C(N,\nu)$ maxima form the input of a L_{-p_2} element, which produces an estimate of the minimum of its inputs.

Obviously, this implementation has an implementation cost, which comes in the form of the number of L_{p_1} elements required. Some implementation optimizations can be performed, however. For example the $\nu C(N,\nu)$ raise-to-a-power elements required in the evaluation of maxima, can be reduced to 1, by moving them to the network input. Figure 4 depicts a median sorting network of this type. It is evident that the number of nonlinear amplifiers has decreased. It is also possible to choose $p_1 \ll p_2$ in order to reduce the implementation complexity of the L_{p_1} elements.

4.2. Simulation results

In order to measure errors introduced by this median approximation network, the following sorting network types were utilized, where $N \in \{3,5,9\}$:

I The odd-even transposition network of Figure 1(a), which was composed of L_p comparators (Figure 1(c)).

II Median approximation network with $p_1 = p_2 = p$.

III Median approximation network with $p_1 = p,\ p_2 = 2p$.

The input set was the same one used in the previous experiments. The MSE was used here as a quantitative criterion. For the network I, only the output that corresponds to the median estimation was taken into account. Results are summarized in Table 1. It is evident that networks II and III reduce error significantly.

5. CONCLUSIONS

This paper introduced the nonlinear mean L_p comparator, an element that can be utilized in sorting network implementations, to improve performance. Although L_p elements return only approximations of the extrema of their inputs, in certain cases the errors introduced may be within acceptable margins. L_p elements have also the added benefit of faster performance, due to the fact that they can be implemented entirely with analog technology.

A design was also proposed that can be utilized in the estimation of the median of a N-element set in a constant number of steps. This approach can be implemented in hardware, where most of the internal operations required can be performed in parallel.

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N	p	I	II	III
3	2	19.956	17.295	17.103
	5	9.766	8.744	7.165
	8	6.352	5.936	4.215
5	2	32.951	20.086	17.468
	5	19.337	16.483	8.106
	8	12.940	13.447	4.787
9	2	35.247	22.444	18.866
	5	24.683	20.763	16.793
	8	18.163	31.173	12.569

Table 1. Comparison of MSE in the approximation of the median of a N-element set.

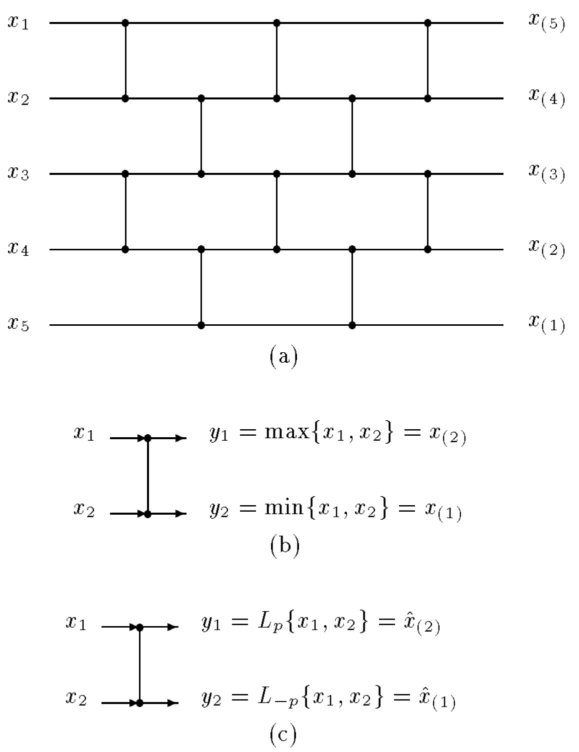

Figure 1. (a) An odd-even transposition sorting network of $N = 5$ inputs. (b) A max/min comparator. (c) A L_p comparator.

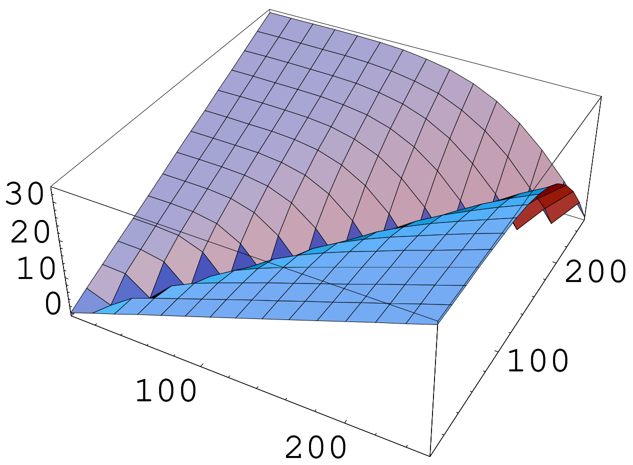

Figure 2. Absolute value of $e_{\max}$ when $p = 5$.

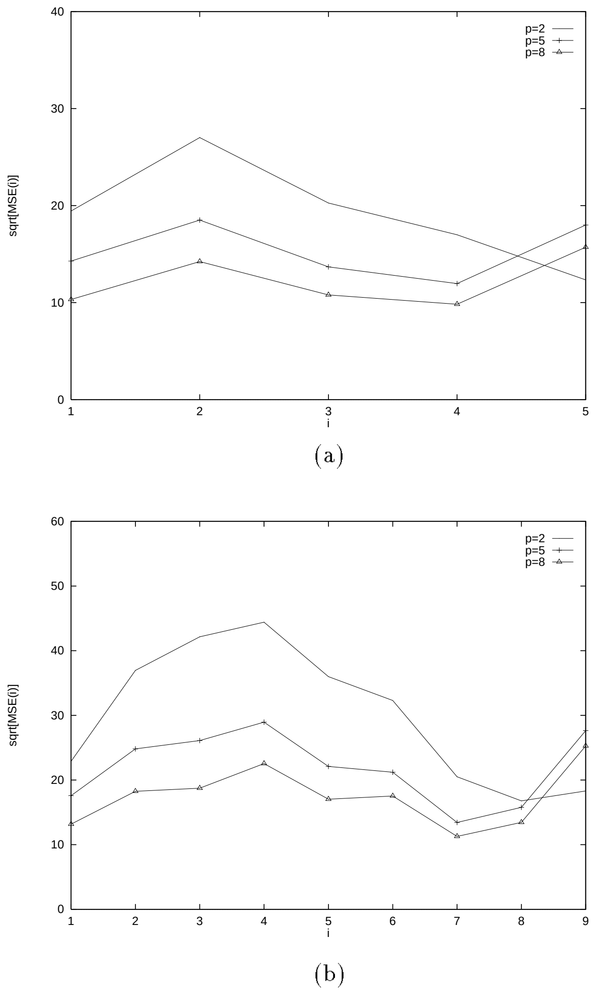

Figure 3. Square root of MSE in modified L_p comparator based sorting networks of (a) $N = 5$ and (b) $N = 9$ inputs.

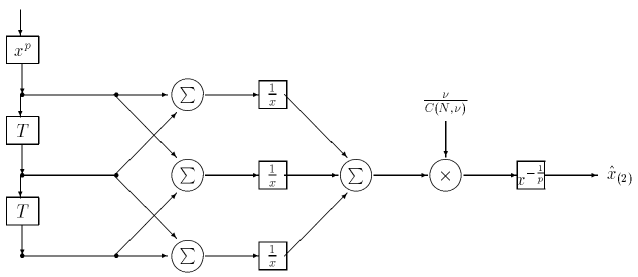

Figure 4. Median approximation sorting network, which utilizes delay elements.