SOFT MORPHOLOGICAL OPERATORS BASED ON NONLINEAR L_P MEAN OPERATORS

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Abstract. The classical grayscale morphological operators erosion and dilation are characterized by strong nonlinearities, a feature that is not desirable in certain image filtering applications. This paper investigates the use of a subclass of nonlinear mean filters, instead of the classical morphological transformation. The nonlinearities of these filters can be hardened or softened progressively, by means of a controlling parameter. The statistical properties of these "soft" morphological filters are investigated. It will be shown that the nonlinear mean filters behave better than or equally well to the grayscale morphological transformations, for certain types of noise.

Key words: mathematical morphology, soft morphology, nonlinear mean filters, L_p mean filters, morphological filtering, L_p morphology

1. Introduction

Lately, alternatives to the grayscale morphological transformations, erosion and dilation, have been studied [1]. One of the reasons that led research this way is the fact that grayscale morphology filters use hard nonlinearities. Therefore their performance in additive noise removal is not always good. Thus, "soft" morphological operators have been proposed. A class of "soft" morphological operators can be based on the use of nonlinear mean filters [2].

The L_p filters, a special category of the general class of nonlinear mean filters, possess some important properties, notably the ability to apply "soft" nonlinearities to a signal, that can be hardened progressively, by choosing a filter parameter appropriately [3]. Thus, L_p filters can be a "soft" substitute of classical max/min operators. The purpose of this paper is to investigate the statistical properties of these filters, as well as their performance under various types of noise.

2. Notations and definitions

Let x_i be an one-dimensional sequence, $A = \{-N, -N+1, \ldots, N-1, N\}$, and a_i, $i \in A$ be the filter coefficients. The L_p *dilation* operator is defined as follows [3] :

$$D_{p,A}(x_i) = \left(\frac{\sum_{j \in A} a_j x_{i-j}^p}{\sum_{j \in A} a_j} \right)^{1/p} \tag{1}$$

where $p \neq 0$. For $p > 0$, L_p dilation is essentially an L_p filter [2]. Similarly the L_p *erosion* can be defined as [3] :

$$E_{p,A}(x_i) = \left(\frac{\sum_{j \in A} a_j x_{i-j}^{-p}}{\sum_{j \in A} a_j} \right)^{-1/p} \tag{2}$$

For $p > 0$, L_p erosion is essentially the L_{-p} filter [2].

It has been shown that the output of the grayscale morphological erosion operator, by a flat structuring element, is always lower than the output of the L_p erosion operator, when $a_j = 1$, for every j inside the filter window A [3, 4]. Similarly, the output of the morphological dilation operator, also by a flat structuring element, is always higher than the output of the L_p dilation operator. For large positive values of p, the output of the L_p erosion and dilation converges to the output of the minimum and maximum operators, respectively. Thus, L_p erosion and dilation are "soft" morphological operators. They can be hardened progressively by choosing the parameter p appropriately.

The counterparts of the grayscale opening and closing are defined, using L_p dilation and erosion, in the following way [3] :

$$y_i = D_{p,A}(E_{p,A}(x_i)) = \left(\sum_{j \in A} \frac{a_j}{\sum_{k \in A} a_k x_{i-j-k}^{-p}} \right)^{1/p} \tag{3}$$

$$y_i = E_{p,A}(D_{p,A}(x_i)) = \left(\sum_{j \in A} \frac{a_j}{\sum_{k \in A} a_k x_{i-j-k}^{p}} \right)^{-1/p} \tag{4}$$

It should be noted that the effective filter window used in the opening/closing definitions is not A but $A' = \{-2N, -2N+1, \ldots, 2N-1, 2N\}$, having size $4N+1$ elements instead of $2N+1$.

3. Statistical properties of L_p opening and closing filters

Based on the statistical properties of the L_p dilation and erosion operators, as analyzed in [2], we shall study the statistical properties of the L_p openings and closings. If we denote by $y_i = f(x_i) = D_{p,A}(E_{p,A}(x_i))$ the output of a L_p opening filter, where $A = \{1, \ldots, N\}$, an approximation of the output mean m_y and the output variance σ_y^2 is given by :

$$m_y \simeq f(x_i)|_{x_{i-j}=m_{x_{i-j}}} + \frac{1}{2} \sum_{j \in A'} f''_{i-j} \sigma^2_{x_{i-j}} + \sum_{\substack{j,\, k \in A' \\ j>k}} f''_{i-j,\, i-k} R(i-j, i-k) \tag{5}$$

$$\sigma_y^2 \simeq \sum_{j \in A'} f_{i-j}^{'2} \sigma_{x_{i-j}}^2 + 2 \sum_{\substack{j,\ k \in A' \\ j>k}} f'_{i-j} f'_{i-k} R(i-j, i-k) - \left[\frac{1}{2} \sum_{j \in A'} f''_{i-j} \sigma_{x_{i-j}}^2 + \sum_{\substack{j,\ k \in A' \\ j>k}} f''_{i-j,i-k} R(i-j, i-k) \right]^2 \quad (6)$$

where $A' = \{2, \ldots, 2N\}$ is the equivalent filter window of this L_p opening operator, $m_{x_{i-j}}$ and $\sigma_{x_{i-j}}^2$ are the mean and the variance of the input sequence at time index $i-j$, respectively, $R(i,k)$ is the auto-correlation function and :

$$f'_{i-j} = \left. \frac{\partial y_i}{\partial x_{i-j}} \right|_{x_{i-j}=m_{x_{i-j}}} \quad (7)$$

$$f''_{i-j} = \left. \frac{\partial^2 y_i}{\partial x_{i-j}^2} \right|_{x_{i-j}=m_{x_{i-j}}} \quad (8)$$

$$f''_{i-j,i-k} = \left. \frac{\partial^2 y_i}{\partial x_{i-j} \partial x_{i-k}} \right|_{x_{i-j}=m_{x_{i-j}}, x_{i-k}=m_{x_{i-k}}} \quad (9)$$

Equations (5)-(6) are valid, if the variances $\sigma_{x_{i-j}}^2$, $j \in A'$ are small, and the probability distributions $p(x_{i-j})$ are short-tailed or medium-tailed.

The first and second derivatives, with respect to x_{i-j}, have the following form :

$$\frac{\partial y_i}{\partial x_{i-j}} = y_i^{1-p} x_{i-j}^{-p-1} \sum_k \frac{a_k a_{j-k}}{\left(\sum_{n \in A} a_n x_{i-n-k}^{-p} \right)^2} \quad (10)$$

$$\frac{\partial^2 y_i}{\partial x_{i-j}^2} = (1-p) y_i^{1-2p} \left[\sum_{l \in A} \frac{a_l \sum_k \delta(j-l-k) a_k x_{i-k-l}^{-p-1}}{\left(\sum_{k \in A} a_k x_{i-l-k}^{-p} \right)^2} \right]^2 - (p+1) y_i^{1-p} \sum_{l \in A} \frac{a_l \left(\sum_{k \in A} a_k x_{i-l-k}^{-p} \right)^2 \sum_{k \in A} \delta(j-l-k) a_k x_{i-l-k}^{-p-2}}{\left(\sum_{k \in A} a_k x_{i-l-k}^{-p} \right)^4} + 2p y_i^{1-p} \sum_{l \in A} \frac{a_l \left(\sum_{k \in A} a_k x_{i-l-k}^{-p} \right) \left(\sum_{k \in A} \delta(j-l-k) a_k x_{i-l-k}^{-p-1} \right)^2}{\left(\sum_{k \in A} a_k x_{i-l-k}^{-p} \right)^4} \quad (11)$$

Let us suppose that an input signal of constant value m_x is corrupted by white zero-mean short-tailed or medium-tailed noise of variance σ_x^2 and the filter coefficients are $a_i = 1,\ i \in A$. In this case, approximations of the variance σ_y^2 and the mean m_y of the L_p opening filtered sequence can be derived by substituting (10) and (11) into (5) and (6), respectively :

$$m_y \simeq m_x - \frac{(3N^3 - 4N^2 + 1)p + 3N^3 - 2N^2 - 1}{6N^3 m_x}\sigma_x^2 \tag{12}$$

$$\frac{\sigma_y^2}{\sigma_x^2} \simeq \frac{2N^2+1}{3N^3} - \left[\frac{(-3N^3 + 4N^2 - 1)p - 3N^3 + 2N^2 + 1}{6N^3 m_x}\right]^2 \sigma_x^2 \tag{13}$$

We can make the following remarks, regarding equations (12) and (13) :

1. The results are valid for white short-tailed or medium-tailed noise, as described above. This means that if the Taylor series approximation is not valid, the theoretical ratio of the variances σ_y^2/σ_x^2 will not correlate well with experimental results. If the noise is not white, equations (5) and (6) should be used instead.
2. We can see from equation (12) that the output mean m_y of the L_p opening and closing filters, is linearly related to the power p. Furthermore, the sign of p does not necessarily imply that a L_p opened (closed) noisy sequence will have negative (positive) bias, when applied to constant regions of the original sequence. This depends on the size of the filter window, i.e. if $p \geq (3N^3 - 2N^2 - 1)/(-3N^3 + 4N^2 - 1)$, the filtered output will be negatively biased, and positively biased otherwise, assuming a filter window $A = \{1, \ldots, N\}$. For example, if $N = 3$, then the bias will change sign at $p_{cng} \simeq -1.3478261$.

4. Simulations

The first set of experiments tests the ability of the L_p morphological operators to remove impulsive and/or additive white noise from 1-dimensional signals. The test signals used were, originally, either constant, having value $m_x = 100$ and length 1000 samples, or sharp edges, of length 20 samples, having values 50 and 100, on either side of the edge, respectively. For the first test signal, the performance of the operators examined, reflects the ability of these operators to remove noise from nearly-constant or slow-varying regions of a signal. For the second test signal, similar conclusions can be extracted, regarding their ability to preserve edges, a characteristic that is very desirable in digital image processing. In each case, the original signals were corrupted by the following types of noise :

I Impulsive noise, consisting of negative impulses only, of height 1, and with probability of occurrence 0.1.

II Positive impulses only, of height 199 and probability of occurrence 0.1.

III Mixed impulsive noise, consisting of both positive and negative impulses, alternating in appearance, with each kind having a probability of occurrence 0.05 and heights 199 and 1 respectively.

IV White additive zero-mean uniform noise, having variance 75.

V White additive zero-mean uniform noise, having variance 300.

VI White additive zero-mean Gaussian noise, having variance 75.

VII White additive zero-mean Gaussian noise, having variance 300.
VIII White additive zero-mean Laplacian noise, having variance 75.
IX White additive zero-mean Laplacian noise, having variance 300.

In the first test, the input signal was the corrupted (originally constant) signal. Subsequently, grayscale opening and closing by a flat structuring set $\{-1, 0, 1\}$, as well as $D_{p,A}(E_{p,A}(\cdot))$ opening and $E_{p,A}(D_{p,A}(\cdot))$ closing operators, with $A = \{-1, 0, 1\}$, were used to filter the noisy sequence. In the case of L_p operators, p took values in the set $\{2, 4, 6\}$ and the filter coefficients were $a_i = 1$, for every i in A and 0 otherwise. It should be clear that the overall window size is 5 points (instead of 3).

The *Noise Reduction Index* (*NRI*), in dB, was used, as a quantitative criterion of filter performance :

$$\begin{aligned} NRI &= 10 \log_{10} \frac{E\left[(s_i - x_i)^2\right]}{E\left[(s_i - y_i)^2\right]} \\ &\simeq 10 \log_{10} \frac{1/N \sum_{i=1}^{N} (s_i - x_i)^2}{1/N \sum_{i=1}^{N} (s_i - y_i)^2} \end{aligned}$$

where s_i, x_i and y_i are the original, noisy and filtered sequences, respectively, and N is the total number of samples per sequence. Results are summarized in Table I. We can notice that, as p increases, the L_p operators non-linearities "harden" progressively and, for large p, converge to the performance of the classical morphological operators. This verifies the theoretical conclusions of Section 3. For certain values of p, L_p opening and closing can have a better NRI by as much as 6 dB, compared to their morphological counterparts. We can also notice that the NRI performance of the L_p operators as a function of p, might have more than one local extrema. For example, we can see in column III that the NRI performance increases, as p increases till $p = -2$, where approximately a local maximum is recorded. Then it falls down to 1.16 dB NRI, for $p = 2$ and starts increasing again, possibly converging to the value of 2.3 dB NRI, for large values of p. This property can lead to the development of adaptive L_p operators.

The second test was conducted using the sharp edge described above, as the original sequence, corrupted by noise of type I-IX. The experiment was repeated fifty times for each noise type and the results were averaged to get a better estimate of the NRI. We used the same filter types that were used in the first test. Table II summarizes the results, where all figures are in NRIs (in dB). Under all noise models, L_p operators performed much better that their classical morphological counterparts, with the exception of the positive-only or negative-only impulsive noise model. This could indicate that L_p operators preserve edges better that their classical morphological counterparts, in the presence of additive white noise.

The ability of the L_p openings and closings to remove both impulsive and additive, white, zero-mean Gaussian noise from two-dimensional signals, was also tested. The reference image used was the 256×256 image "Lenna". The following noise models were used :

I Mixed impulsive noise, consisting of both positive and negative impulses, alternating in appearance with each kind having a probability of occurrence 0.05 and heights 255 and 0.001 respectively.

II White additive zero-mean Gaussian noise, having variance 100.

III White additive zero-mean Gaussian noise, having variance 100 and mixed impulsive noise of consisting of both positive and negative impulses of height 255 and 0.001, respectively, having equal probabilities of appearance equal to 0.05.

The corrupted image was filtered by both classical and L_p openings and closings, where p took values in the set $\{2, 5, 8\}$. Each operation (e.g. erosion or dilation) employed a square 3×3 filter window $A = \{a_{ij} : i,\ j \in \{-1, 0, 1\}\}$. In the case of L_p filtering, the filter coefficients were $a_{ij} = 1$, for every i, j inside the filter window A, and zero otherwise. The NRI (in dB) has been used as a quantitative performance criterion. Results are displayed in Table III. In the case of mixed impulsive noise and/or Gaussian noise, L_p operators perform much better, recording a NRI difference of at least 3 dB, compared to their morphological counterparts. However, as p increases the performance of the L_p operators converges gradually to the performance of the respective grayscale operators, as expected.

5. Conclusions

This paper presented some important statistical properties of the L_p operators. The power parameter p controls the strength of the nonlinearity of the L_p operators, when applied to a signal. Their performance was shown to be superior to the performance of classical morphological operators, in most of the noise models examined. Specifically, for certain values of the parameter p, and for impulsive and additive zero-mean white short-tailed and medium-tailed noise, the performance difference was significant. This fact suggests that filtering approaches which adapt both the power parameter p and the number and types of L_p basical operators (e.g. erosion, dilation) in cascade, could prove to be very effective.

References

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TABLE I
NRI performance in dB, of various filters operating on a corrupted (originally constant) signal, using a 5-point window.

Method	I	II	III	IV	V	VI	VII	VIII	IX
Grayscale Closing	**14.55**	-1.87	2.86	1.13	0.93	1.39	1.53	1.09	2.42
L_6 Closing	13.06	0.83	5.59	5.68	4.69	5.86	5.59	5.67	5.81
L_4 Closing	11.80	1.84	6.51	5.84	5.21	6.10	6.24	6.15	6.20
L_2 Closing	8.39	3.61	**8.07**	**5.90**	**5.57**	**6.20**	**6.71**	**6.52**	**6.24**
L_2 Opening	-1.98	9.33	1.16	5.69	5.13	5.79	5.59	6.37	3.34
L_4 Opening	-1.38	11.46	1.92	5.44	4.49	5.37	4.64	5.95	3.04
L_6 Opening	-1.22	**12.23**	2.11	5.13	3.86	4.94	3.97	5.51	2.76
Grayscale Opening	-1.07	10.61	2.30	0.70	0.93	1.20	1.35	2.23	1.12

TABLE II
NRI performance in dB, of various filters operating on a noisy sharp edge, using a 5-point window.

Method	I	II	III	IV	V	VI	VII	VIII	IX
Grayscale Closing	**14.25**	-0.69	0.72	0.96	0.57	0.72	1.22	0.71	1.21
L_6 Closing	12.06	1.25	2.81	**3.56**	3.45	**3.11**	4.21	**2.79**	3.74
L_4 Closing	10.56	2.04	3.68	3.09	3.92	2.78	4.80	2.44	4.28
L_2 Closing	7.54	3.74	**5.58**	2.23	**4.16**	2.13	**5.15**	1.84	**4.82**
L_2 Opening	-2.24	10.88	3.24	0.92	2.97	1.23	3.02	1.05	2.72
L_4 Opening	-1.62	12.37	3.94	1.24	2.63	1.64	2.75	1.53	2.53
L_6 Opening	-1.43	**12.90**	4.16	1.62	2.38	2.06	2.46	2.09	2.38
Grayscale Opening	-1.24	12.00	4.40	0.22	0.79	1.02	0.98	1.42	1.31

TABLE III
NRI performance in dB of both "hard" (min/max) and "soft" (L_p) morphological filtering schemes.

Method	I	II	III
Grayscale Opening	-0.71	-2.01	-0.85
L_8 Opening	-1.32	0.77	-0.86
L_5 Opening	-1.67	0.72	-1.25
L_2 Opening	-3.74	0.74	-3.21
L_2 Closing	**11.32**	1.27	**11.21**
L_5 Closing	9.19	**1.50**	8.70
L_8 Closing	7.99	1.48	7.25
Grayscale Closing	4.93	-2.14	3.26