1. INTRODUCTION

The binary (or "scalar") morphological operators erosion and opening, as well as their dual counterparts dilation and closing, have been used successfully, to remove positive-only or negative-only impulsive noise from binary images. However, their performance in binary images corrupted by inversion noise is not so good. Additionally, in the case of multiple instances (versions) of a binary image, corrupted by the same type of noise, these operators can not exploit the strong correlation between these instances. Such a case appears when we have multiple noisy instances of the same image and we want to produce on output image only.

The binary matrix set morphological operators, that will be presented in this paper, represent a simple, yet powerful, extension to the scalar morphological operators. They have the capability to treat a number of binary images as a single entity. As an application, the noise filtering problem, presented in the previous paragraph, can be formulated easily, in matrix set terms, with performance that surpasses the one of scalar operators.

2. DEFINITIONS

If A and B are two binary sets, belonging to the same Euclidean space E, then the *classical ("scalar") opening* A_B and *closing* A^B of the set A, by the structuring set B, is given by :

$$A_B \stackrel{\text{def}}{=} (A \ominus B) \oplus B \tag{1}$$

$$A^B \stackrel{\text{def}}{=} (A \oplus B) \ominus B \tag{2}$$

where [1]:

$$A \oplus B = \bigcup_{b \in B} A_b \tag{3}$$

$$A \ominus B = \bigcap_{b \in B} A_b \tag{4}$$

A matrix $\mathbf{A}$, of size $I \times J$, of binary sets, can be written as [2]:

$$\mathbf{A} = \{A_{ij} \subseteq E : \ i = 1, \ldots, I, \ j = 1, \ldots, J\} \tag{5}$$

Definition (5) can be written in a less strict form as :

$$\mathbf{A} = \{A_{ij}\} \tag{6}$$

Let $\mathbf{B}$ be a $J \times K$ binary matrix set, of the form of equation (5). The *binary matrix set dilation* $\mathbf{A} \oplus \mathbf{B}$ and *erosion* $\mathbf{A} \ominus \mathbf{B}$, of the matrix set $\mathbf{A}$, by the *structuring matrix* $\mathbf{B}$, is

a $I \times K$ matrix set, which is given by the following equations :

$$\mathbf{A} \oplus \mathbf{B} \stackrel{\text{def}}{=} \{C_{ik} \subseteq E : \; C_{ik} = \bigcap_{j=1}^{J} A_{ij} \oplus B_{jk}, \; i = 1, \ldots, I, \; k = 1, \ldots, K\} \tag{7}$$

$$\mathbf{A} \ominus \mathbf{B} \stackrel{\text{def}}{=} \{C_{ik} \subseteq E : \; C_{ik} = \bigcup_{j=1}^{J} A_{ij} \ominus B_{jk}, \; i = 1, \ldots, I, \; k = 1, \ldots, K\} \tag{8}$$

The *matrix set opening* $\mathbf{A_B}$ and *closing* $\mathbf{A^B}$, of the matrix set $\mathbf{A}$, by the structuring matrix $\mathbf{B}$, can be defined in terms of equations (7)-(8), as follows :

$$\mathbf{A_B} \stackrel{\text{def}}{=} (\mathbf{A} \ominus \mathbf{B}) \oplus \mathbf{B}^T \tag{9}$$

$$\mathbf{A^B} \stackrel{\text{def}}{=} (\mathbf{A} \oplus \mathbf{B}) \ominus \mathbf{B}^T \tag{10}$$

The definitions of the matrix morphological operators, given by equations (7)-(10), greatly resemble the definitions of matrix morphological operators as presented in [2, 3]. However, in those definitions, the roles of set union and intersection are reversed, e.g. in the case of matrix set dilation, we use the set *intersection* of dilated sets, while matrix set dilation is defined as the set *union* of dilated sets in [2, 3]. These differences arise from the fact that, in each case, the matrix operators are formulated with dissimilar goals.
Essentially, the scalar opening and closing can be viewed as special cases of the matrix set opening and closing, respectively.

3. PROPERTIES

We have proven several properties of the matrix set operators. They possess only a subset of the general properties of their scalar counterparts. For example, the matrix dilation and erosion operators are dual morphological transformations, for which the increasing property holds. However, many distributive properties do not hold. Some of the general properties of the matrix set operators, are the following :

Property 1 Duality :

$$\mathbf{A} \oplus \mathbf{B} = (\mathbf{A}^c \ominus \mathbf{B})^c \tag{11}$$

$$\mathbf{A} \ominus \mathbf{B} = (\mathbf{A}^c \oplus \mathbf{B})^c \tag{12}$$

$$\mathbf{A_B} = \left((\mathbf{A}^c)^{\mathbf{B}}\right)^c \tag{13}$$

$$\mathbf{A^B} = ((\mathbf{A}^c)_{\mathbf{B}})^c \tag{14}$$

Property 2 Increasing :

$$\mathbf{A} \subseteq \mathbf{B} \Rightarrow \begin{cases} \mathbf{A} \oplus \mathbf{X} \subseteq \mathbf{B} \oplus \mathbf{X} \\ \mathbf{A} \ominus \mathbf{X} \subseteq \mathbf{B} \ominus \mathbf{X} \\ \mathbf{A_X} \subseteq \mathbf{B_X} \\ \mathbf{A^X} \subseteq \mathbf{B^X} \end{cases} \tag{15}$$

Properties (1)-(2) are easy to prove. Thus, their proofs are omitted.

Property 3 *Let* $\mathbf{X}$ *be a* $1 \times M$ *binary matrix set. Let* $\mathbf{B}$ *be a* $M \times N$ *binary matrix set, which has identical elements, that is :*

$$\mathbf{B} = \{B_{mn} = B\} \tag{16}$$

In this case, the matrix opening $\mathbf{X_B}$ *is equal to :*

$$\mathbf{X_B} = \left[\left(\bigcup_{m=1}^{M} X_{1m} \ominus B\right) \oplus B \cdots \left(\bigcup_{m=1}^{M} X_{1m} \ominus B\right)\right] \tag{17}$$

$$= \left[\bigcup_{m=1}^{M} (X_{1m})_B \cdots \bigcup_{m=1}^{M} (X_{1m})_B\right] \tag{18}$$

A similar property holds for matrix closing.

Proof: Property (3) is easy to prove. By noticing that $X \ominus B$ is equal to :

$$\mathbf{X} \ominus \mathbf{B} = [X_{11}\ X_{12}\ \cdots\ X_{1M}] \ominus \begin{bmatrix} B & \cdots & B \\ \vdots & \ddots & \vdots \\ B & \cdots & B \end{bmatrix}$$

$$\mathbf{X} \ominus \mathbf{B} = \left[\bigcup_{m=1}^{M} (X_{1m} \ominus B) \cdots \bigcup_{m=1}^{M} (X_{1m} \ominus B)\right] \tag{19}$$

and by substituting (19) into (9), property (3) can be easily shown to hold.

4. APPLICATIONS

The form of the matrix opening given by equation (17) hints that a fast implementation is feasible. Initially, we can erode each one of the M noisy instances X_{1m}, by the structuring element B. It is obvious that since these operations are independent, they can be performed concurrently, in one step. Subsequently, the union of the M resulting sets produces the output. Dilation of the resulting matrix yields one, having all elements the same.

We can use binary matrix morphology in several applications (e.g. multiframe binary image noise filtering). Let us suppose that we have M corrupted *instances* (or *versions*) of the same *reference* binary image. We will assume that, for all instances, the corruption type is the same, modelled by *inversion* (*not*) noise. Our goal will be to extract *one* binary image, which, hopefully, will resemble the reference image, utilising all M corrupted instances (in the filtering process). A straightforward way to attain this goal, is to perform scalar opening (or closing) on a per-instance basis. Subsequently, the M opened (or closed) images could be combined into one, e.g. by using a median filter, with window size M, operating pointwise. This is the classical scalar approach (*Method I*). Another approach, which utilises better the strong correlation between the noisy instances, is to form a $1 \times M$ matrix set $\mathbf{A}$, with the M noisy instances as elements. We could perform a matrix opening $\mathbf{A_B}$ (or closing $\mathbf{A^B}$), where by $\mathbf{B}$ we denote a $M \times M$ structuring matrix, having elements either the same, or different (*Method II*). According to property (3), the $\mathbf{Y} = \mathbf{A_B}$ ($\mathbf{Y} = \mathbf{A^B}$ respectively) binary matrix set will have identical elements.

In general, if the elements of the structuring matrix $\mathbf{B}$ are not identical, then the sets Y_{1m}, where $m = 1, \ldots, M$, will be different. In this case, we can use a median filter, operating pointwise, with a filter window of M samples, to produce one output binary image (*Method III*). Results are summarised in Table 1, for $M = 3$. The *Mean Square Error (MSE)* was used as a quantitative criterion of filter performance, defined as :

$$MSE = E[(s-y)^2] \simeq \frac{\sum_{k=1}^{K}\sum_{l=1}^{L}(s_{kl}-y_{kl})^2}{NP} \tag{20}$$

where s_{kl}, y_{kl} are the values of the reference and filtered images, respectively, at the point (k, l) and K, L are the image dimensions. The reference image was the thresholded version of "Lenna". Methods I and II utilised 3×3 "rhombus" structuring elements, while method III also utilised single point structuring elements, centered at the origin, as the non-diagonal structuring matrix elements. It is evident that the matrix opening outperforms the scalar opening in every case.

The morphological matrix set operators can, also, be used in filtering binary image *sequences*, corrupted by inversion (not) noise. We shall assume that the spatial information does not vary rapidly between successive frames. Let $\mathbf{X}$ denote a $1 \times (2M+1)$ matrix set, which consists of $2M+1$ successive noisy frames. Thus, the matrix set will have the form $\mathbf{X} = \left[X_{1(m-M)} \cdots X_{1i} \cdots X_{1(m+M)}\right]$, where X_{1m} is the noisy frame at time index m. We can create an estimate of the reference frame, at time index m, by performing a matrix opening (or closing) of $\mathbf{X}$ with a structuring matrix of the form of equation (16). It is obvious that this method exploits well the strong temporal correlation of the image sequence. The performance of the matrix morphological operators was compared with the performance of a *3-D median filter*. The filter window consisted of $||W|| = (2M+1)||B||$ samples, where the operator $||\cdot||$ denotes the number of points belonging to a set. In simulations, both the matrix and the 3-D median filters outperformed the scalar-based approaches. The performance of the 3-D median filter was similar to the performance of the matrix operators. However, the matrix opening and closing, using a structuring matrix with identical elements, can be implemented very fast, as was shown previously.

5. CONCLUSIONS

This paper presents an extension of the binary scalar morphological operators, the so called binary matrix morphological operators. It was shown that their performance is very good, compared to the performance of their scalar counterparts, in multiframe inversion noise filtering problems. Additionally, when structuring matrices are used which have identical elements, a fast parallel implementation of the matrix opening is possible.

REFERENCES

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Table 1: MSE performance of scalar and matrix opening filtering of binary images, corrupted by either 10% or 30% inversion noise. Lower figures indicate better performance.

Method	*10% noise*	*30% noise*
Method I	0.0438	0.2734
Method II	0.0217	0.1191
Method III	0.0211	0.1985