Grayscale Morphology using Nonlinear L_p Mean Filters

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ABSTRACT — **The use of grayscale morphological filters in image processing is well known. This paper investigates the use of a class of nonlinear mean filters instead of the classical morphological transformations, erosion and dilation. The statistical properties of these filters are investigated. For reference purposes, their performance is also compared to the performance of their grayscale counterparts. It is shown that the nonlinear mean filters behave better than or equally well to the grayscale morphological transformations, in various types of noise examined.**

1 Introduction

Mathematical morphology is a wide research area that gives methods, useful in many aspects of image and signal processing [1]. Lately, alternatives to the grayscale morphological transformations, erosion and dilation, have been studied [2]. One of the reasons that led research this way is the fact that grayscale morphology filters use hard non-linearities. Therefore their performance in additive noise removal is not always good. Thus "soft" morphological operators have been proposed. Another class of "soft" morphological operators can be based on the use of nonlinear mean filters [3].

The L_p filters, a special category of the general class of nonlinear mean filters, possess some important properties, notably the ability to apply "soft" non-linearities to a signal, that can be hardened progressively, by choosing the parameter p appropriately. Thus, L_p filters can be a "soft" substitute of classical max/min operators. The purpose of this paper is to investigate some general properties of these filters, when used to create "soft" morphological filters.

2 Definitions

Let x_i be an one-dimensional sequence, $A = \{-N, -N+1, \ldots, N-1, N\}$, and a_i, $i \in A$ be the filter coefficients. We can define the L_p dilation as follows :

$$D_{p,A}(x_i) = \left(\frac{\sum_{j \in A} a_j x_{i-j}^p}{\sum_{j \in A} a_j} \right)^{1/p} \quad (1)$$

where $p \neq 0$. For $p > 0$, L_p dilation is essentially an L_p filter [3]. Similarly we can define the L_p erosion as :

$$E_{p,A}(x_i) = \left(\frac{\sum_{j \in A} a_j x_{i-j}^{-p}}{\sum_{j \in A} a_j} \right)^{-1/p} \quad (2)$$

For $p > 0$, L_p erosion is essentially the L_{-p} filter [3]. Since the following equation holds [4] :

$$\begin{aligned} &\min\{x_{i-j}, j \in A\} \leq E_{p,A}(x_i) \leq \\ &\leq D_{p,A}(x_i) \leq \max\{x_{i-j}, j \in A\} \end{aligned} \quad (3)$$

we can come to the conclusion that the output of the grayscale morphological erosion, with a flat structuring element, is always lower than the output of the L_p erosion, when $a_j = 1$, for every j inside the filter window A. Similarly, the output of the morphological dilation, also with a flat structuring element, is always higher than the output of the L_p dilation. For large positive values of p, the output of the L_p erosion and dilation converges to the output of the minimum and maximum operators, respectively. Thus L_p erosion and dilation are "soft" morphological operators. They can be hardened progressively by choosing the parameter p appropriately.

The counterparts of the grayscale opening and closing can be defined, using L_p dilation and erosion, in the following way :

$$\begin{aligned} y_i &= D_{p,A}(E_{p,A}(x_i)) = \\ &= \left(\sum_{j \in A} \frac{a_j}{\sum_{k \in A} a_k x_{i-j-k}^{-p}} \right)^{1/p} \end{aligned} \quad (4)$$

$$y_i = E_{p,A}(D_{p,A}(x_i)) = \left(\sum_{j \in A} \frac{a_j}{\sum_{k \in A} a_k x_{i-j-k}^p} \right)^{-1/p} \quad (5)$$

It should be noted that the effective filter window used in the opening/closing definitions is not A but $A' = \{-2N, -2N+1, \ldots, 2N-1, 2N\}$, having size $4N+1$ elements, instead of $2N+1$.

3 Properties

3.1 General properties

The L_p operators possess some interesting properties which are listed below, where the proofs are omitted :

Property 1 *Duality :*

$$D_{-p,A}(x_i) = E_{p,A}(x_i) \quad (6)$$

$$D_{p,A}(x_i^{-1}) = E_{p,A}^{-1}(x_i) \quad (7)$$

$$D_{p,A}(E_{p,A}(x_i)) = \left(E_{p,A}(D_{p,A}(x_i^{-1}))\right)^{-1} \quad (8)$$

Property 2 *If x_i and y_i are two sequences with positive terms, satisfying $x_i \leq y_i$ for every i, and the filter coefficients a_j are non-negative, then the following inequalities hold :*

$$D_{p,A}(x_i) \leq D_{p,A}(y_i) \quad (9)$$

$$E_{p,A}(x_i) \leq E_{p,A}(y_i) \quad (10)$$

$$E_{p,A}(D_{p,A}(x_i)) \leq E_{p,A}(D_{p,A}(y_i)) \quad (11)$$

$$D_{p,A}(E_{p,A}(x_i)) \leq D_{p,A}(E_{p,A}(y_i)) \quad (12)$$

Property 3 *Let $D_{p,A_i}(\cdot)$ be m L_p dilation filters connected in a cascade, where $A_i = \{N_i, \ldots, M_i\}$, $i \in \{1, \ldots, m\}$. The application of this filter on an input signal x_i, having non-negative terms, is equivalent to one application of a $D_{p,B}(\cdot)$ dilation filter having filter window $B = \{\sum_{i=1}^m N_i, \ldots, \sum_{i=1}^m M_i\}$ and filter coefficients b_j, $j \in B$ satisfying :*

$$\frac{b_j}{\sum_{k \in B} b_k} = \frac{a_j^{(1)} * a_j^{(2)} * \cdots * a_j^{(m)}}{\prod_{i=1}^m \sum_{n \in A_i} a_n^{(i)}}, \ \forall j \in B \quad (13)$$

where $a_j^{(i)}$ is the jth coefficient of window A_i and $a_j^{(i)} = 0$, $j \notin A$.

3.2 Statistical properties

The statistical properties of the L_p dilation and erosion, have been analyzed in [3]. Based on the conclusions of this paper, we can derive the statistical properties of the L_p openings and closings. It can be shown that, if we use a $D_{p,A}(E_{p,A}(\cdot))$ opening operator to filter an originally constant one-dimensional sequence, which is subsequently corrupted by additive white, zero-mean short-tailed or medium-tailed noise, then the output mean value is given, approximately, by :

$$m_y \simeq m_x - \frac{(3N^3-4N^2+1)p+3N^3-2N^2-1}{6N^3 m_x} \sigma_x^2 \quad (14)$$

where m_x, m_y are the mean of the corrupted and the filtered sequence respectively, σ_x^2 is the variance of the corrupted sequence and the filter window coefficients are $a_i = 1$, $i \in A$, where $A = \{1, \ldots, N\}$.

From (14) we can note that the sign of the power p does not necessarily imply that a L_p opened (closed) noisy sequence will have negative (positive) bias, when applied to constant regions of the original sequence. This depends on the size of the filter window N as well, as it is evident from (14). In contrast, the operator used in classical grayscale morphological operations, defines the sign of the bias of the filtered sequence.

3.3 Mixed impulsive noise filtering

As discussed in [3], since impulsive noise has a long-tailed distribution, the results of the previous section do not hold. However, some other interesting properties can be derived in this case.

Let us suppose that a signal of constant value m, is corrupted by mixed impulsive noise of total probability q, having equal positive and negative impulse probability $q/2$. We will also assume that the positive and negative impulses have heights cm, $c \gg 1$ and dm, $d \ll 1$ respectively. The filtering of the resulting input sequence x_i will be considered successful when the following inequality holds for the filtered output y_i :

$$am \leq y_i \leq bm, \ a = 1-\epsilon, \ b = 1+\epsilon, \ \epsilon \geq 0 \quad (15)$$

Let us suppose that a L_p opening or closing filter is used, having window coefficients $a_j = 1$, $j \in A = \{1, \ldots, N\}$. It can be proven that the following inequality holds, in the case of L_p opening :

$$\frac{b^{-p}-1}{\frac{c^{-p}+d^{-p}}{2}-1} \leq q \leq \frac{a^{-p}-1}{\frac{c^{-p}+d^{-p}}{2}-1} \quad (16)$$

and in the case of L_p closing respectively :

$$\frac{a^p - 1}{\frac{c^p + d^p}{2} - 1} \leq q \leq \frac{b^p - 1}{\frac{c^p + d^p}{2} - 1} \qquad (17)$$

Essentially, for given a, b, c, d and p, (16) and (17) define upper and lower bounds for the impulse probability q. L_p opening and closing filters can suppress impulses whose probability lies only in the intervals given by (16), (17) respectively (of course, by taking into account that noise suppression is defined by (15)). Also, by comparing (16), (17) we can come to the conclusion that similar performance between opening and closing is expected, when the following equalities hold :

$$c = d^{-1} \qquad (18)$$

$$a = b^{-1} \qquad (19)$$

For the rest of this paper, unless otherwise stated, a filter window $A = \{-1, 0, 1\}$ will be employed *for each morphological operation* with window coefficients $a_j = 1$, $j \in A$ and 0 otherwise.

4 Simulations

In order to test the validity of (14), a test signal of constant value $m = 100$ and length $N = 2000$ samples, was corrupted by additive white zero-mean uniform noise, of variance $\sigma^2 = 75$. Subsequently, the corrupted sequence was fed to either a $D_{p,A}(E_{p,A}(\cdot))$ opening or $E_{p,A}(D_{p,A}(\cdot))$ closing filter, with coefficients $a_j = 1$, $j \in A = \{-1, 0, 1\}$. An approximation $\hat{m}_y$ of the output mean was calculated, while the theoretical output mean m_y, for the conditions of each test run, was calculated directly from (14). The results are summarised in Table 1. It is evident that the theoretical results of Section 3.2 correlate very well with the experimental ones.

Simulation results indicate that the L_p opening and closing operators perform better than their classical counterparts, in the *MSE* sense, in nearly-constant regions of a signal. Simulations were performed using a test signal of value 100, which was subsequently corrupted by the following types of noise :

I Impulsive noise, consisting of negative impulses only, of height 1, and with probability of occurrence 0.1.

II Positive impulses only, of height 199 and probability of occurrence 0.1.

III Mixed impulsive noise, consisting of both positive and negative impulses, alternating in appearance, with each kind having a probability of occurrence 0.05 and heights 199 and 1 respectively.

IV White additive zero-mean uniform noise, having variance 75.

V White additive zero-mean Gaussian noise, having variance 75.

The corrupted sequences were filtered by both classical and L_p opening and closing operators, which utilised an effective filter window of 5 samples. The *Noise Reduction Index (NRI)* quantitative criterion in dB, was utilised :

$$NRI \simeq 10 \log_{10} \frac{\sum_i (s_i - x_i)^2}{\sum_i (s_i - y_i)^2} \qquad (20)$$

where s_i, x_i, y_i are the values of the original, corrupted and filtered sequences at time index i, respectively. Results are summarised in Table 2. It is evident that L_p filter performance, in the case of mixed impulsive noise, is better than that of the classical morphological operators. The classical operators perform slightly better for impulsive noise constisting of negative-only or positive-only spikes. However, it should be noted that the performance of a L_p operator can match the performance of the respective classical operator, by choosing the parameter p appropriately. In the case of rapidly-varying regions of a signal, classical operators preserve the discontinuities better that the L_p operators. Once again, by setting the parameter p high enough, the L_p operators can follow discontinuities satisfactorily. At the same time, the filtered signal bias is much smaller, for L_p morphological filters, than the one imposed by classical morphological filtering.

Regarding the theoretical results of Section 3.3, if we substitute $a = 0.9$, $b = 1.1$, $c = 1.99$ and $d = 0.01$ into (15), we can note that simulation results (as depicted in column III of Table 2) correlate well with the theoretical results of Section 3.3. For example, if the filtering is performed by a L_2 closing operator, then the maximum output impulse occurrence probability $q_{\max}$ given by (17) is 0.21, a figure which correlates well with the experimental NRI performance of 8.07 dB.

The ability of the L_p openings and closings to remove both impulsive and additive, white, zero-mean Gaussian noise from two-dimensional signals, was also tested. The reference image used was the 256×256 image "Lenna". The following noise models were used :

I Mixed impulsive noise, consisting of both positive and negative impulses, alternating in appearance with each kind having a probability of occurrence 0.05 and heights 255 and 0.001 respectively.

II White additive zero-mean Gaussian noise, having variance 100.

III White additive zero-mean Gaussian noise, having variance 100 and mixed impulsive noise of consisting of both positive and negative impulses of height 255 and 0.001, respectively, having equal probabilities of appearance equal to 0.05.

The corrupted image was filtered by both classical and L_p openings and closings, where p took values in the set $\{2, 5, 8\}$. Each operation (e.g. erosion or dilation) employed a filter window $A = \{(a_i, a_j) : i, j \in \{-1, 0, 1\}\}$. In the case of L_p filtering, the filter coefficients were $a_{ij} = 1$, for every i, j inside the filter window A, and zero otherwise. The NRI, in dB, has been used, as a quantitative performance criterion. Results are displayed in Table 3. In the case of mixed impulsive noise and/or Gaussian noise, L_p operators perform much better, recording a NRI difference of at least 3 dB, compared to their morphological counterparts. However, as p increases the performance of the L_p operators converges gradually to the performance of the respective grayscale operators.

5 Conclusions

L_p "soft" morphological operators possess many interesting properties. Most of them resemble properties of their respective classical morphological operator counterparts. L_p operators also possess a unique property that could lead to the development of fast implementations.

Their performance was shown to be superior to the performance of classical morphological operators, in most of the noise model cases examined. We must stress again that the performance of L_p operators is mainly dependent on the value of the parameter p. We have seen that for certain values of the parameter p, and for impulsive and additive zero-mean white short-tailed and medium-tailed noise, L_p operators perform much better that the classical morphological ones. This fact suggests that filtering approaches which adapt both the power parameter p *and* the number and types of L_p basical operators (e.g. erosion, dilation) in cascade, could prove to be very effective.

References

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Table 1: Comparison of the theoretical and the computed output mean of a L_p filtered sequence.

Method	*Theoretical*	*Simulation*
L_6 Closing	100.991	100.882
L_4 Closing	100.565	100.520
L_2 Closing	100.139	100.137
L_2 Opening	99.287	99.351
L_4 Opening	98.861	98.975
L_6 Opening	98.435	98.626

Table 2: NRI performance in dB, when filtering various types of noise, using a 5-point effective window.

Method	I	II	III	IV	V
Graysc. Cl.	**14.55**	-1.87	2.86	1.13	1.39
L_6 Closing	13.06	0.83	5.59	5.68	5.86
L_4 Closing	11.80	1.84	6.51	5.84	6.10
L_2 Closing	8.39	3.61	**8.07**	**5.90**	**6.20**
L_2 Opening	-1.98	9.33	1.16	5.69	5.79
L_4 Opening	-1.38	11.46	1.92	5.44	5.37
L_6 Opening	-1.22	**12.23**	2.11	5.13	4.94
Graysc. Op.	-1.07	10.61	2.30	0.70	1.20

Table 3: NRI performance in dB of both "hard" (min/max) and "soft" (L_p) morphological filtering schemes.

Method	I	II	III
Grayscale Opening	-0.71	-2.01	-0.85
L_8 Opening	-1.32	0.77	-0.86
L_5 Opening	-1.67	0.72	-1.25
L_2 Opening	-3.74	0.74	-3.21
L_2 Closing	**11.32**	1.27	**11.21**
L_5 Closing	9.19	**1.50**	8.70
L_8 Closing	7.99	1.48	7.25
Grayscale Closing	4.93	-2.14	3.26