THEORETICAL ANALYSIS OF L_P MEAN COMPARATORS

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ABSTRACT

In certain signal processing applications there is a need for fast implementations of sorting networks. Digital implementations of sorting networks that rely on a Digital Signal Processor core are not as efficient as their analog counterparts. This paper deals with the L_p comparators for which simple analog implementations exist that employ operational amplifier-based adders, logarithmic and antilogarithmic multipliers. From a statistical point of view, the L_p comparators are based on the nonlinear means. Their probability density function and the first and second-order moments are derived for independent uniformly distributed inputs. L_p comparators provide estimates of the minimum and maximum of their inputs. Therefore, they introduce errors. A proper approach to compensate for the estimation errors is proposed.

1. INTRODUCTION

Sorting is a fundamental operation in data processing. Sorting operations are estimated to account for over 25% of processing time for all computations [1]. Sorting is the basic operation employed in order statistics filters that constitute effective techniques for image/signal processing due to their robustness properties. Besides signal processing, other numerous applications of sorting can be found, e.g., in database management, communication networks, multiaccess memory, multiprocessors, shared disks. Sorting algorithms have been extensively explored in the past few decades. Sorting networks are special cases of sorting algorithms. A sorting network has N inputs $x_1, \ldots, x_N$ and N outputs $x_{(1)}, \ldots, x_{(N)}$, where $x_{(i)}$ denotes the i-th order statistic of the set $\{x_1, \ldots, x_N\}$. That is, $x_{(1)}$ denotes the smallest element of the set, while $x_{(N)}$ denotes the largest element. Two of the most commonly used algorithms is the odd-even transposition network [10] and the Batcher's bitonic sorter [2]. The basic functional unit of a sorting network is the comparator, which receives two numbers at its inputs and presents their maximum and minimum at its outputs. Although sorting networks based on other functional units have been proposed, e.g., sorting networks based on a three-element median [4], the most common type of sorting network employs comparators. Using comparators, many authors have presented sorting networks of different sizes [5], while others have examined the fault tolerance of this kind of sorting networks [6, 7]. Recently, a sorting network is shown to be a wave digital filter realization of an N-port memoryless nonlinear classical network [8].

The order statistics filters employ usually a Digital Signal Processor core. However, sorting is a computationally expensive operation, and a large area and power reduction can be obtained with simpler analog implementations [3]. This paper deals with the theoretical properties of the L_p comparator. Sorting networks based on L_p comparators were first proposed in [9]. However, L_p comparators are "noisy" comparators. Therefore, we have to compensate for the errors that are introduced by the L_p comparators, before we replace the conventional comparators in a sorting network with the proposed L_p comparators. To devise such an error compensation algorithm, first the statistical properties of the L_p comparators are explored and compared against those of the min-max comparators. Then, we propose a simple error compensation algorithm and we derive theoretically the gain that is obtained, when L_p comparators employing error compensation are used. Accordingly, the present paper extends the previously reported work [9].

The outline of the paper is as follows. The definition of L_p comparators with two inputs is given in Section 2. Their statistical properties are derived in this Section as well. The compensation for the errors that are introduced by the L_p comparators is treated in Section 3. Implementation details are given in Section 4.

2. L_P COMPARATORS

In this section, the L_p comparator is defined and its statistical properties are derived for independent uniformly distributed input samples. The L_p comparator employs nonlinear L_{-p} and L_p means with two inputs to estimate the minimum and maximum of two input samples, respectively [10], i.e.,

$$\hat{x}_{(1)} = L_{-p}(x_1, x_2) = \left(\frac{x_1^{-p} + x_2^{-p}}{2} \right)^{-1/p} \quad (1)$$

$$\hat{x}_{(2)} = L_p(x_1, x_2) = \left(\frac{x_1^{p} + x_2^{p}}{2} \right)^{1/p} \quad (2)$$

where p is a positive real number different than 1, i.e., $p \in I\!R^+ - \{0, 1\}$. In contrast to the classical min/max comparator, whose output is one of input samples, an L_p comparator provides an estimate of the minimum and the maximum sample.

If x_i, $i = 1, 2$, are independent random variables (RVs) uniformly distributed in the interval $[0, L]$, the probability density function (pdf) of the random variable $z = L_p(x_1, x_2)$ is given by:

$$f(z) = \begin{cases} \dfrac{2^{2/p}}{L^2\, p}\, B(1/p, 1/p)\, z & \text{if} \quad 0 \le z < 2^{-1/p}\, L \\ \dfrac{2^{2/p}}{L^2\, p}\, B(1/p, 1/p)\, z \left(2\, I_{\frac{L^p}{2z^p}}(1/p, 1/p) - 1 \right) & \text{if} \quad 2^{-1/p}\, L \le z < L \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

where $B(\cdot)$ denotes the Beta function and $I_x(a,b)$ is the incomplete Beta function defined as [12]:

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1} \, (1-t)^{b-1} \, dt. \tag{4}$$

Proof of (3): Let x_i, $i = 1, 2$ be independent RVs uniformly distributed in the interval $[0, L]$. Then the pdf of the RV $\zeta = x^p$ is given by [11]:

$$f_\zeta(\zeta) = \frac{1}{p\,L} \, \frac{1}{\zeta^{1-1/p}} \quad 0 \le \zeta \le L^p. \tag{5}$$

The pdf of the RV $\xi = \zeta_1 + \zeta_2$, where $\zeta_i = x_i^p$, $i = 1, 2$, is given simply by the convolution of the pdfs $f_{\zeta_1}(\xi)$ and $f_{\zeta_2}(\xi)$ [11]. That is,:

$$f_\xi(\xi) = \begin{cases} 0 & \text{if} \quad \xi < 0 \\ \frac{1}{L^2\,p^2} \int_0^\xi \lambda^{1/p-1} \, (\xi-\lambda)^{1/p-1} \, d\lambda & \\ & \text{if} \quad 0 \le \xi < L^p \\ \frac{1}{L^2\,p^2} \int_{z-L^p}^{L^p} \lambda^{1/p-1} \, (\xi-\lambda)^{1/p-1} \, d\lambda & \\ & \text{if} \quad L^p \le \xi < 2\,L^p \\ 0 & \text{if} \quad \xi \ge 2\,L^p \end{cases} \tag{6}$$

In (6), it is easily seen that [12]:

$$\int_0^\xi \lambda^{1/p-1} \, (\xi-\lambda)^{1/p-1} \, d\lambda = B(1/p, 1/p) \, \xi^{2/p-1} \tag{7}$$

where $B(\cdot)$ denotes the Beta function. To calculate the remaining integral, we apply the change of variable $\lambda = \frac{\xi}{2} + \psi$, i.e.:

$$\int_{\xi/2-L^p}^{L^p-\xi/2} (\frac{\xi}{2}+\psi)^{1/p-1} \, (\frac{\xi}{2}-\psi)^{1/p-1} \, d\psi = \xi^{2/p-1}$$
$$(B(1/p,1/p) - 2\int_0^{1-\frac{L^p}{\xi}} \psi^{1/p-1} (1-\psi)^{1/p-1} \, d\psi).$$

Let z denote the following function of the RV ξ: $z = 2^{-1/p} \, \xi^{1/p}$. Then, the pdf of the RV z is given by [11]:

$$f_z(z) = 2\,p\,z^{p-1} \, f_\xi(2\,z^p) \tag{8}$$

for z such that $2\,z^p$ belongs to the domain of $f_\xi(\cdot)$, which completes the proof.

The pdf of RV z is plotted for $p = 2, 5$ and 8 in Figure 1. For completeness, the pdf of the RV $x_{(2)}$ for uniform parent distribution in the interval $[0, L]$ and $N = 2$ is included:

$$f_{(2)}(x) = \frac{2}{L} \, \frac{x}{L}, \quad 0 \le x \le L. \tag{9}$$

It can be shown that the expected value and the mean square value of the RV z is given by:

$$\mathrm{E}\{z\} = \frac{2\,L}{3} \left(1 - \frac{2^{-1/p}}{p} \int_0^1 t^{1/p} \, (1+t)^{1/p-1} \, dt\right) \tag{10}$$

$$\mathrm{E}\{z^2\} = \frac{L^2}{2} \left(1 - \frac{2^{-2/p}}{p} \int_0^1 t^{1/p} \, (1+t)^{2/p-1} \, dt\right), \tag{11}$$

respectively.

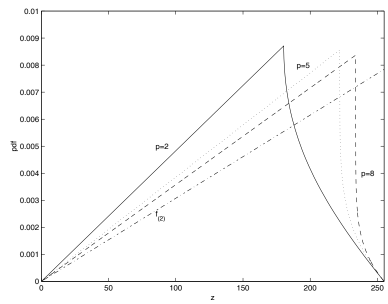


Figure 1: Probability density function of the RV $z = L_p(x_1, x_2)$ for $p = 2, 5$, and 8, when x_1 and x_2 are independent RVs uniformly distributed in the interval $[0, L]$.

Proof of (10):

$$\mathrm{E}\{z\} = \frac{2^{2/p}}{p} \, B(1/p,1/p) \, \frac{L}{3} \, (2^{1-3/p} - 1) + \frac{2^{1+2/p}}{L^2\,p}$$
$$\int_{2^{-1/p}\,L}^{L} z^2 \left[\int_0^{2\,z^p} t^{1/p-1} \, (1-t)^{1/p-1} dt\right] dz. \tag{12}$$

The integral in (12) can be calculated by integration by parts and by applying Leibnitz's rule for the differentiation of the inner integral, i.e.:

$$\int_{2^{-1/p}\,L}^{L} z^2 \left[\int_0^{2\,z^p} t^{1/p-1} \, (1-t)^{1/p-1} dt\right] dz = \frac{1}{3}\Big[L^3$$
$$B(1/p,1/p) \, I_{1/2}(1/p,1/p) \; - 2^{-3/p} \, L^3 B(1/p,1/p)$$
$$I_1(1/p,1/p) + \frac{p\,L^p}{2} \int_{2^{-1/p}\,L}^{L} z^{2-p} \, \left(\frac{L^p}{2\,z^p}\right)^{1/p-1}$$
$$\left(1 - \frac{L^p}{2\,z^p}\right)^{1/p-1} \, dz.\Big] \tag{13}$$

It is well known that [13]:

$$I_{1/2}(1/p,1/p) = 1/2, \qquad I_1(1/p,1/p) = 1 \quad \forall p > 0. \tag{14}$$

By applying the change of variable $t = \frac{L^p}{2\,z^p}$, the last term in (13) is rewritten as:

$$\frac{p\,L^p}{2} \int_{2^{-1/p}\,L}^{L} z^{2-p} \, \left(\frac{L^p}{2\,z^p}\right)^{1/p-1} \, \left(1 - \frac{L^p}{2\,z^p}\right)^{1/p-1} \, dz =$$
$$2^{-3/p} \, L^3 \int_{1/2}^{1} t^{-2/p-1} (1-t)^{1/p-1} \, dt \tag{15}$$

The substitution of (13)–(15) in (12) yields:

$$\mathrm{E}\{z\} = \frac{2\,L}{3\,p} \, 2^{-1/p} \int_{1/2}^{1} t^{-2/p-1} (1-t)^{1/p-1} \, dt \tag{16}$$

that can further be simplified by appropriate variable changes and integration by parts to (10).

The following approximate expressions for the first and second moment of RV z hold:

$$\mathrm{E}\{z\} \approx \frac{L}{2} \left[(1 + 2^{-1/p}) - \frac{B(1/p,1/p)}{6\,p} (3 - 2^{-1/p})\right] \tag{17}$$

$$\mathrm{E}\{z^2\} \approx 2^{-2/p} L^2 \left[2^{1/p} + \frac{B(1/p,1/p)}{2p} \left(\frac{1}{2} - 2^{1/p}\right) \right]. \quad (18)$$

Proof of (17): Let us rewrite (3) as follows:

$$f(z) = \begin{cases} \frac{2^{2/p}}{L^2\, p}\, B(1/p,1/p) z & \text{if } 0 \le z < 2^{-1/p}\, L \\ g(z) & \text{if } 2^{-1/p}\, L \le z < L \\ 0 & \text{otherwise.} \end{cases} \quad (19)$$

The pdf $f(z)$ results from convolution, accordingly it is continuous. From its continuity, we know the exact values of function $g(z)$ at $z = 2^{-1/p}\, L$ and $z = L$, i.e.:

$$g(2^{-1/p}\, L) = \frac{2^{-1/p}}{L\, p}\, B(1/p,1/p) \quad \text{and} \quad g(L) = 0. \quad (20)$$

Moreover, at the same values of the argument z, the cumulative distribution $G(z)$ attains the following values:

$$G(2^{-1/p}\, L) = \frac{1}{2\, p}\, B(1/p,1/p) \quad \text{and} \quad G(L) = 1. \quad (21)$$

The expected value of the RV z is given by:

$$\mathrm{E}\{z\} = \int_0^{2^{-1/p}\, L} \frac{2^{2/p}}{L^2}\, 2\, G(2^{-1/p}\, L)\, z^2\, dz + \int_{2^{-1/p}\, L}^{L} z\, g(z)\, dz. \quad (22)$$

Integrating by parts the second integral in (22), we obtain:

$$\mathrm{E}\{z\} = -2^{-1/p}\, \frac{L}{3}\, G(2^{-1/p}\, L) + L - \int_{2^{-1/p}\, L}^{L} G(z)\, dz. \quad (23)$$

By applying the trapezoidal rule, we obtain the following approximation for the integral in (23):

$$\int_{2^{-1/p}\, L}^{L} G(z)\, dz \simeq \frac{1 + G(2^{-1/p}\, L)}{2}\, (L - 2^{-1/p}\, L). \quad (24)$$

The substitution of (24) in (23) yields:

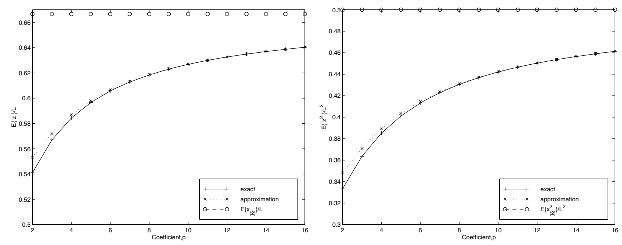
$$\mathrm{E}\{z\} = \frac{L}{2}\left(1 + 2^{-1/p}\right) - \frac{G(2^{-1/p}\, L)\, L}{6}\left(3 - 2^{-1/p}\right). \quad (25)$$

By combining (21) and (25), we obtain (17).
Proof of (18): By following the procedure outlined above, we rewrite $\mathrm{E}\{z^2\}$ as follows:

$$\begin{aligned} \mathrm{E}\{z^2\} \quad &= \quad \int_0^{2^{-1/p}\, L} \frac{2^{2/p}}{L^2}\, 2\, G(2^{-1/p}\, L)\, z^3\, dz \\ &\quad + \int_{2^{-1/p}\, L}^{L} z^2\, g(z)\, dz. \end{aligned} \quad (26)$$

The integration by parts of the second integral in (26), and the application of the trapezoidal rule to approximate the second integral yields (18).

The expected value and the mean square value of the RV z for several values of the coefficient p are plotted in Figure 2(a) and (b), respectively. The approximate values obtained by using (17) and (18) are overlaid for comparison purposes. It is seen that for $p > 8$ the values obtained by the approximate expressions are practically the same to those obtained by numerical integration of (10) and (11). The expressions in (10) and (11) should be compared to those



(a) (b)

Figure 2: First and second moment of the RV $z = L_p(x_1, x_2)$ for several values of the coefficient p. (a) Expected value; (b) Mean square value.

of the order statistics for $N = 2$ and uniform parent distribution that are given by [14]:

$$\mathrm{E}\{x_{(2)}\} = \frac{2\, L}{3} \quad \text{and} \quad \mathrm{E}\{x_{(2)}^2\} = \frac{L^2}{2} \quad (27)$$

It is obvious that the first and second moments of the RV z tend to those of the RV $x_{(2)}$ for large p.

Similarly, if x_i, $i = 1, 2$, are independent RVs uniformly distributed in the interval $[\epsilon, L]$, the pdf of the random variable $w = L_{-p}(x_1, x_2)$ is given by:

$$f_w(w) = \begin{cases} \frac{2^{-2/p}}{p\,(L-\epsilon)^2} w \int_{1-\frac{w^p}{2\,\epsilon^p}}^{\frac{w^p}{2\,\epsilon^p}} t^{-(1+\frac{1}{p})}(1-t)^{-(1+\frac{1}{p})}\, dt \\ \qquad \text{if} \quad \epsilon < w \le 2^{1/p}\, \frac{\epsilon\, L}{(\epsilon^p + L^p)^{1/p}} \\ \frac{2^{-2/p}}{p\,(L-\epsilon)^2} w \int_{\frac{w^p}{2\, L^p}}^{1-\frac{w^p}{2\, L^p}} t^{-(1+\frac{1}{p})}(1-t)^{-(1+\frac{1}{p})}\, dt \\ \qquad \text{if} \quad 2^{1/p}\, \frac{\epsilon\, L}{(\epsilon^p + L^p)^{1/p}} < w \le L \\ 0 \qquad \text{otherwise.} \end{cases} \quad (28)$$

The pdf of the RV $x_{(1)}$ for uniform parent distribution in the interval $[0, L]$ for $N = 2$ is given by:

$$f_{(1)}(x) = \frac{2}{L}\, \left(1 - \frac{x}{L}\right), \quad 0 \le x \le L. \quad (29)$$

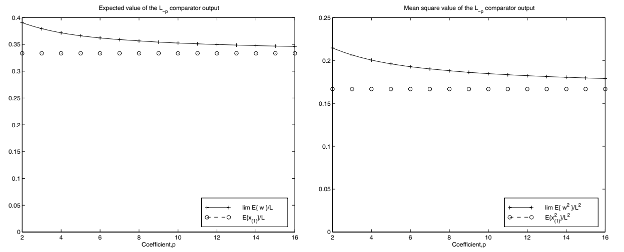
The limit of the expected value and the mean square value of the RV w for $\epsilon \to 0$ is given by:

$$\lim_{\epsilon \to 0} \mathrm{E}\{w\} = \frac{4}{3}\, 2^{1/p-1}\, L \int_0^{2^{-1/p}} \frac{t\, dt}{(1-t^p)^{1+1/p}} \quad (30)$$

$$\lim_{\epsilon \to 0} \mathrm{E}\{w^2\} = 2^{2/p-1}\, L^2 \int_0^{2^{-1/p}} \frac{t^2\, dt}{(1-t^p)^{1+1/p}}, \quad (31)$$

respectively. The expected value and the mean square value of the RV w for several values of the coefficient p are plotted in Figure 3(a) and (b), respectively. It can easily be verified that for p large, the first and the second moment of the RV w approximate those of the RV $x_{(1)}$, i.e.:

$$\mathrm{E}\{x_{(1)}\} = \frac{L}{3} \quad \text{and} \quad \mathrm{E}\{x_{(1)}^2\} = \frac{L^2}{6}. \quad (32)$$



(a) (b)

Figure 3: Limit of first and second moment of the RV $w = L_{-p}(x_1, x_2)$ for several values of the coefficient p when $\epsilon \rightarrow 0$. (a) Expected value; (b) Mean square value.

3. ERROR COMPENSATION

L_p comparators introduce errors. Let $e_{\max}(x_1, x_2) = x_{(2)} - \hat{x}_{(2)}$ denote the error introduced by the L_p comparator in the estimation of the maximum of two input samples. Then:

$$0 \leq e_{\max}(x_1, x_2) \leq \frac{1}{2}\,|x_2 - x_1|. \tag{33}$$

Similarly, let $e_{\min}(x_1, x_2) = x_{(1)} - \hat{x}_{(1)}$ denote the corresponding error in the estimation of the minimum of two input samples. It can easily be shown that:

$$-\frac{1}{2}\,|x_2 - x_1| \leq e_{\min}(x_1, x_2) \leq 0. \tag{34}$$

For $x_i,\ i = 1, 2$, independent RVs uniformly distributed in the interval $[0, L]$ it can be shown that the mean squared error (MSE) introduced by the L_p comparator is given by:

$$\mathrm{E}\{e^2_{\max}(x_1, x_2)\} = \frac{L^2}{2}\,\left\{1 + 2^{-2/p}\int_0^1 (1+t^p)^{2/p}\,dt - 2^{1-1/p}\int_0^1 (1+t^p)^{1/p}\,dt\right\}. \tag{35}$$

Proof of (35): Let $x_i,\ i = 1, 2$, be independent uniformly distributed RVs in the interval $[0, L]$. Let also $s = x_2 - x_1$ denote their difference. Then, the MSE between the L_p comparator and the maximum sample $x_{(2)}$ among the input samples x_1 and x_2 is given by:

$$\mathrm{E}\{e^2_{\max}(x_1, x_2)\} = \int_0^L dx_1\ f_{x_1}(x_1)\int_{s<0}\left[x_1 - \frac{(x_1^p + (x_1+s)^p)^{1/p}}{2^{1/p}}\right]^2 f_s(s|x_1)\ ds\ + \int_0^L dx_1\ f_{x_1}(x_1) \int_{s\geq 0}\left[(x_1+s) - \frac{(x_1^p + (x_1+s)^p)^{1/p}}{2^{1/p}}\right]^2 f_s(s|x_1)\ ds. \tag{36}$$

It can easily be proven that:

$$f_s(s|x_1) = \begin{cases} \frac{1}{L} & -x_1 \leq s \leq L - x_1 \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

By using (37), (36) is rewritten:

$$\mathrm{E}\{e^2_{\max}(x_1, x_2)\} = \frac{1}{L^2}\,\Big(\int_0^L dx_1 \int_{-x_1}^0 \Big[x_1 - \frac{(x_1^p + (x_1+s)^p)^{1/p}}{2^{1/p}}\Big]^2 ds + \int_0^L dx_1 \int_0^{L-x_1}\Big[(x_1+s) - \frac{(x_1^p + (x_1+s)^p)^{1/p}}{2^{1/p}}\Big]^2 ds\Big). \tag{38}$$

By evaluating all integrals in (38) and invoking L' Hospital's rule:

$$\lim_{x\rightarrow 0} x^4 \int_1^{L/x} t\,(1+t^p)\,dt = 0. \tag{39}$$

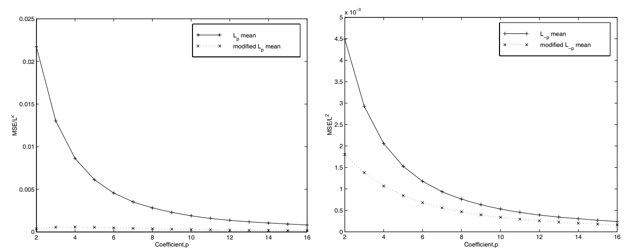
we obtain (35).

The MSE of the L_p comparator is plotted for several values of the coefficient p in Figure 4(a). It is seen that the larger the coefficient p is, the smaller the MSE introduced by the L_p comparator becomes. Accordingly, for large values of the coefficient p, the L_p comparator converges to the max operator, as expected.

If $x_i,\ i = 1, 2$, are independent RVs uniformly distributed in the interval $[\epsilon, L]$, it can be shown that for $\epsilon \rightarrow 0$, the limit of the MSE of the L_{-p} comparator is:

$$\lim_{\epsilon\rightarrow 0}\mathrm{E}\{e^2_{\min}(x_1, x_2)\} = L^2\,\left\{\frac{1}{6} + 2^{2/p-1}\int_0^1 \frac{t^2}{(1+t^p)^{2/p}}\,dt - 2^{1/p}\int_0^1 \frac{t^2}{(1+t^p)^{1/p}}\,dt\right\}. \tag{40}$$

The MSE of the L_{-p} comparator is plotted for several values of the coefficient p in Figure 4(b) as well. It is seen that the larger the coefficient p is, the smaller the MSE introduced by the L_{-p} comparator becomes. Accordingly, for large values of the coefficient p, the L_{-p} comparator converges to the min operator, as expected.



(a) (b)

Figure 4: (a) MSE of the L_p comparator and the modified L_p comparator in estimating the maximum of two independent input samples that are uniformly distributed in the interval $[0, L]$. (b) Limit of the MSE of the L_{-p} comparator and the modified L_{-p} comparator in estimating the minimum of two independent input samples that are uniformly distributed in the interval $[\epsilon, L]$, for $\epsilon \rightarrow 0$.

Next, we compensate for the MSE introduced by the L_p comparators for small p. We argue that the estimation error increases almost linearly with the absolute value of the difference between x_1, x_2 (i.e., their distance). Accordingly, we propose to modify the L_p comparator outputs as follows:

$$\tilde{x}_{(1)} = L_{-p}(x_1, x_2) - d\,|s|, \quad d > 0 \tag{41}$$

$$\tilde{x}_{(2)} = L_p(x_1, x_2) + c\,|s|, \quad c > 0 \tag{42}$$

where $s = x_2 - x_1$ and c and d are constants. The constants c and d can be chosen so that the $\mathrm{E}\{\tilde{e}_{\max}^2\}$ and $\lim_{\epsilon \to 0} \mathrm{E}\{\tilde{e}_{\min}^2\}$ is minimized, respectively. It can be shown that the optimal constants c and d are given by:

$$c = \frac{3}{2} + 3 \cdot 2^{-1/p} \int_0^1 (s-1)\,(1+s^p)^{1/p}\, ds \tag{43}$$

$$d = -\frac{1}{2} - 3 \cdot 2^{1/p} \int_0^1 \frac{s\,(s-1)}{(1+s^p)^{1/p}}\, ds, \tag{44}$$

respectively. The optimal constants c and d are plotted for several values of the coefficient p in Figure 5. The MSE between the mod-

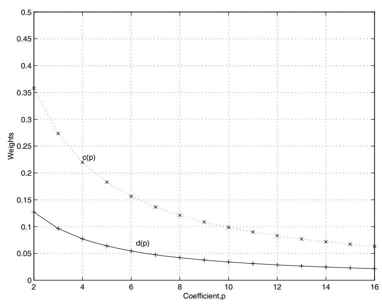


Figure 5: Optimal constants c and d that minimize the MSE between the modified L_p comparator output and the true maximum and minimum of two independent uniformly distributed samples, for several values of the coefficient p.

ified L_p comparator output (42) and the true maximum sample is given by:

$$\mathrm{E}\{\tilde{e}_{\max}^2\} = \mathrm{E}\{e_{\max}^2\} - \frac{c^2\, L^2}{6}. \tag{45}$$

It is overlaid in Figure 4(a) for comparison purposes. Similarly, the MSE between the modified L_{-p} comparator output (41) and the true minimum sample is given by:

$$\mathrm{E}\{\tilde{e}_{\min}^2\} = \mathrm{E}\{e_{\min}^2\} - \frac{d^2\, L^2}{6}. \tag{46}$$

It is shown overlaid in Figure 4(b).

4. IMPLEMENTATION OF L_P COMPARATORS

The basic module in the analog implementation of the L_p comparator is the so-called *multifunction converter* [15, pp. 113-116]. A multifunction converter consists of four operational amplifiers, four logging transistors and four resistors. The exponent p is determined by two external resistors. Raising to an arbitrary power p and computing the p-th root can be achieved with the same module by controlling an external potentiometer together with two fixed resistors. Accuracy 0.2% can be achieved for p ranging from 0.2 to 5 [15]. This module can be used to raise to a power p, to compute p-th roots and as a divider. The latter operation is need in the implementation of L_{-p} comparator. Indeed, an L_{-p} comparator involves the inversion of the input signals, the computation of the L_p mean of the inverted input signals, and finally, the inversion of the L_p output. Moreover, the same module can be used to implement the correction term in (41)-(42). Clearly, the absolute value can be computed with a cascade of an adder and a multifunction converter that can be used first to raise to an even power (e.g., 2 or 4) and then to compute the p-th root. Accordingly, efficient pipelined architectures for L_p comparators of two inputs can be developed to estimate the minimum and maximum by exploiting an adder, a multifunction converter used to raise to the power p and compute the p-th root, and a divider. The preceding theoretical analysis yields an efficient error compensation for small values of p (e.g., 2) enabling the practical use of L_p comparators, because accurate performance of multifunction converters are guaranteed for small values of p.

5. REFERENCES

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