EIGENVALUE DISTRIBUTION OF THE CORRELATION MATRIX IN $L\ell$-FILTERS

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ABSTRACT

$L\ell$-filters are a fundamental filter class within the family of order statistic filters. In this paper, starting from first principles, we derive the cumulative density function and the probability density function of time and ranked ordered samples for independent identically/non-identically distributed input random variables. The raw second moments and the product moments of the time and ranked ordered samples are then computed for independent identically distributed input samples. Based on the aforementioned moments the correlation matrix of the time and ranked ordered samples is derived and its eigenvalue distribution is determined. We present relationships between the eigenvalues of the correlation matrix of time and ranked ordered samples and those of the correlation matrix of the ordered samples.

1. INTRODUCTION

Nonlinear filters have become a very important tool in signal processing, and especially in image analysis and computer vision. For a review of the nonlinear filter classes the reader may consult [1]. One of the best known nonlinear families is based on the order statistics. It uses the concept of data ordering. One of the major classes of order statistic filters is the $L\ell$-filter [2, 3]. It has the form of a linear combination of the observations and it exploits the combined rank and location information inherent in the observations. Powerful extensions of the $L\ell$-filters have been proposed in the literature, such as the order statistic filter banks [4], the permutation filter lattices [5]. Closely related estimators to the $L\ell$-estimator have been proposed for the estimation of the mean using order statistics in [6]. In practice, adaptive designs based on the Least Mean Squares algorithm (LMS) of $L\ell$-filters and their extensions prevail [7].

In this paper, starting from first principles, we derive the cumulative density function and the probability density function of time and ranked ordered samples for independent identically/non-identically distributed input random variables. The raw second moments and the product moments of the time and ranked ordered samples are then computed for independent identically distributed (i.i.d.) input samples. Based on the aforementioned moments the correlation matrix of the time and ranked ordered samples is derived and its eigenvalue distribution is determined. Although $L\ell$-filters loose some of their advantage in an i.i.d. environment, like permutation filters [8], because all time-rank orderings are equally likely, the properties derived under this assumption give a valuable insight into the operation of adaptive LMS $L\ell$-filters in respect of their convergence in the mean and in the mean-square. Such an i.i.d. environment is the case of a constant signal corrupted by additive zero-mean white noise. For an observation vector of length N, we prove that N out of the N^2 eigenvalues of the time and ranked ordered samples are given by the eigenvalues of the correlation matrix of the ranked ordered samples divided by N. We also derive lower and upper bounds for the minimal and maximal eigenvalue of the correlation matrix of the time and ranked ordered samples. These bounds depend on the extreme eigenvalues of the correlation matrix of the ranked ordered samples. The latter eigenvalues are related through inequalities with the eigenvalues of the ranked ordered noise samples. The theoretical results have been verified by numerical computations.

The outline of the paper is as follows. Section 2 briefly describes $L\ell$-estimators. Section 3 deals with the probability density function of one time and ranked ordered sample and the joint probability density function of two time and ranked ordered samples. The eigenvalues of the correlation matrix of independent identically distributed time and ranked ordered samples are studied in Section 4. Upper and lower bounds on the extreme eigenvalues are derived in this section as well. Similar bounds on the extreme eigenvalues of the correlation matrix of ranked ordered samples are presented in Section 5.

2. $L\ell$-ESTIMATORS

Let us consider that the observed signal $\{x(n)\}$ can be expressed as a sum of an arbitrary noise-free signal $\{s(n)\}$ plus zero-mean additive white noise $\{v(n)\}$, where n denotes discrete-time. Let $\mathbf{x}_\ell(n)$ be the $N \times 1$ vector of input observations

$$\mathbf{x}_\ell(n) = (x_1(n), x_2(n), \ldots, x_N(n))^T \tag{1}$$

and $\mathbf{x}_L(n)$ be the ranked ordered input vector at n given by

$$\mathbf{x}_L(n) = \left(x_{(1)}(n), x_{(2)}(n), \ldots, x_{(N)}(n)\right)^T \tag{2}$$

where $x_{(1)}(n) \leq x_{(2)}(n) \leq \ldots \leq x_{(N)}(n)$. The vector $\mathbf{x}_L(n)$ is commonly referred as the vector of the *order statistics* of $\mathbf{x}_\ell(n)$. Let us define the *temporal location vector* [5]

$$\boldsymbol{\xi}_{(i)}(n) = (\xi_{i1}(n), \xi_{i2}(n), \ldots, \xi_{iN}(n))^T \tag{3}$$

where

$$\xi_{ij}(n) = \begin{cases} 1 & \text{if } x_{(i)}(n) \leftrightarrow x_j(n) \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

In (4) $x_{(i)}(n) \leftrightarrow x_j(n)$ denotes that the ith order statistic occupies the jth temporal sample. By using the temporal location vector we create the following $N^2 \times 1$ vector

$$\mathbf{x}_{L\ell}(n) \quad = \quad \left(x_{(1)}(n)\boldsymbol{\xi}^T_{(1)}(n) \mid x_{(2)}(n)\boldsymbol{\xi}^T_{(2)}(n) \mid \cdots \mid x_{(N)}(n)\right.$$

$$\boldsymbol{\xi}^T_{(N)}(n)\big)^T = \big(x_{(1)1}(n), \ldots, x_{(1)N}(n) \mid \ldots \mid x_{(N)1}(n), \ldots, x_{(N)N}(n)\big)^T. \quad (5)$$

An estimate $\widehat{s}(n)$ of the original (noise-free) signal can be obtained by

$$\widehat{s}(n) = \mathbf{c}^T \mathbf{x}_{L\ell}(n) \quad (6)$$

that defines the so-called N^2-*$L\ell$-estimator* [2]. Henceforth we call this estimator $L\ell$-estimator for brevity.

Let us assume that $\mathbf{x}_{L\ell}(n)$ defined in (5) and $s(n)$ are jointly stationary stochastic signals. Based on the stationarity assumption we are interested in the derivation of the N^2-$L\ell$-filter that minimizes the mean squared error

$$\begin{aligned}\varepsilon &= \mathrm{E}\{(s(n) - \widehat{s}(n))^2\} \\ &= \mathbf{c}^T \mathbf{R}_{L\ell}\, \mathbf{c} - 2\mathbf{c}^T \mathbf{p}_{L\ell} + \mathrm{E}\{s^2(n)\}\end{aligned} \quad (7)$$

where $\mathbf{R}_{L\ell} = \mathrm{E}\{\mathbf{x}_{L\ell}(n)\mathbf{x}^T_{L\ell}(n)\}$ is the correlation matrix of the time and ranked ordered samples, $\mathbf{x}_{L\ell}(n)$, and $\mathbf{p}_{L\ell} = \mathrm{E}\{s(n)\, \mathbf{x}_{L\ell}(n)\}$ is the cross-correlation vector between the vector $\mathbf{x}_{L\ell}(n)$ and the desired response $s(n)$. Clearly, provided that $\mathbf{R}_{L\ell}$ is not singular, the optimal N^2-$L\ell$-filter coefficient vector $\mathbf{c}^o$ is given by

$$\mathbf{c}^o = \mathbf{R}^{-1}_{L\ell} \mathbf{p}_{L\ell}. \quad (8)$$

Approaches to determining the $L\ell$-filter coefficients $\mathbf{c}$ using the LMS algorithm have been proposed in [3, 9]. The convergence in the mean and in the mean square of the LMS-based design approaches depends strongly on the eigenvalue distribution of $\mathbf{R}_{L\ell}$ [10]. The objective of this paper is to study the eigenvalue distribution of $\mathbf{R}_{L\ell}$ in an i.i.d. environment.

3. PROBABILITY DENSITY FUNCTION OF THE CORRELATION MATRIX OF TIME AND RANKED ORDERED SAMPLES

Let $g_i(t)$ and $G_i(t)$ denote, respectively, the probability density function (pdf) and the cumulative density (cdf) function the i-th input observation. The pdf of the random variable $x_{(m)i}$ for independent non-identically distributed observations is given by the following expression:

$$\begin{aligned} f_{(m)i}(t) &= g_i(t) \sum_{\substack{(n_1, n_2, \ldots, n_{m-1}) \\ n_1, n_2, \ldots, n_{m-1} \in \mathcal{S}_i^N \\ n_1 < n_2 < \ldots < n_{m-1}}} G_{n_1}(t) G_{n_2}(t) \ldots G_{n_{m-1}}(t) \\ & \prod_{\substack{n_l \\ n_l \in \mathcal{S}_{i, n_1, \ldots, n_{m-1}}}} [1 - G_{n_l}(t)] \\ & m = 1, 2, \ldots, N \quad \text{and} \quad i = 1, 2, \ldots, N \end{aligned} \quad (9)$$

where $\mathcal{S} = \{1, 2, \ldots, N\}$, $\mathcal{S}_i^N = \mathcal{S} - \{i\}$, and $\mathcal{S}_{i, n_1, n_2, \ldots, n_{m-1}} = \mathcal{S}_i^N - \{n_1, n_2, \ldots, n_{m-1}\}$. In (9) the summation extends over all permutations $(n_1, n_2, \ldots, n_{m-1})$ of $1, 2, \ldots, i-1, i+1, \ldots, N$ which are $\binom{N-1}{m-1}$ in total. Eq. (9) can be proved starting from first principles for $N = 3$, and subsequently applying mathematical induction. Alternatively, it can be obtained as a special case of the analysis in [8, 11]. For $N = 3$ and $m = i = 1$ (9) yields

$$f_{(1)\,1}(t) = g_1(t)\, [1 - G_2(t)]\, [1 - G_3(t)]\,. \quad (10)$$

The cdf of the random variable $x_{(m)i}$ can be obtained by

$$F_{(m)i}(t) = \int_{-\infty}^{t} f_{(m)i}(\psi) d\psi. \quad (11)$$

For i.i.d. input observations, we have

$$\begin{aligned} f_{(m)i}(t) &= \binom{N-1}{m-1} g(t) G^{m-1}(t) [1 - G^{N-m}(t)] \\ &= \frac{1}{N} f_{(m)}(t) \end{aligned} \quad (12)$$

where $f_{(m)}(t)$ is the pdf of the m-th order statistic $x_{(m)}$ [12].

For $1 \le k < l \le N$, $i, j = 1, 2, \ldots, N$, $i \ne j$ and $t_1 < t_2$ the joint pdf of the random variables $x_{(k)i}$ and $x_{(l)j}$ is given by

$$\begin{aligned} f_{(k)i\,(l)j}(t_1, t_2) &= g_i(t_1) g_j(t_2) \sum_{\substack{(q_1, q_2, \ldots, q_{k-1}) \\ q_1, q_2, \ldots, q_{k-1} \in \mathcal{S}_{ij}^N \\ q_1 < q_2 < \ldots < q_{k-1}}} \\ & \sum_{\substack{(n_1, n_2, \ldots, n_{l-k-1}) \\ n_1, n_2, \ldots, n_{l-k-1} \in \mathcal{S}_{i,j,q_1,\ldots,q_{k-1}} \\ n_1 < n_2 < \ldots < n_{l-k-1}}} \prod_{\rho=1}^{k} G_{q_\rho}(t_1) \prod_{m=1}^{l-k-1} [G_{n_m}(t_2) - \\ & -G_{n_m}(t_1)] \prod_{\tau \in \mathcal{S}_{i,j,q_1,\ldots,q_{k-1},n_1,\ldots,n_{l-k-1}}} [1 - G_\tau(t_2)]\,. \end{aligned} \quad (13)$$

For i.i.d. input observations (13) yields

$$f_{(k)i\,(l)j}(t_1, t_2) = \begin{cases} \frac{1}{N\,(N-1)} f_{(k)(l)}(t_1, t_2) & \text{if} \quad t_1 < t_2 \\ \frac{1}{N} f_{(l)}(t_2) & \text{if} \quad t_1 \ge t_2 \end{cases} \quad (14)$$

where $f_{(k)(l)}(t_1, t_2)$ is the joint pdf of the order statistics $x_{(k)}$ and $x_{(l)}$.

4. EIGENVALUES OF THE CORRELATION MATRIX OF TIME AND RANKED ORDERED SAMPLES

In the subsequent analysis we assume that the input observations are i.i.d. random variables, i.e., the noise-free signal is a constant s. Under this assumption, using (12) and (14), we obtain the following expressions for the second-order moment of the random variable $x_{(m)i}$ and the product moments of the random variables $x_{(k)i}$ and $x_{(l)j}$:

$$\mathrm{E}\{x^2_{(k)i}\} = \frac{1}{N} \mathrm{E}\{x^2_{(k)}\} \quad (15)$$

$$\mathrm{E}\{x_{(k)i} x_{(l)j}\} = \frac{1}{N\,(N-1)} \mathrm{E}\{x_{(k)} x_{(l)}\} \quad (16)$$

where $k < l$, $i \ne j$ and $k, l = 1, 2, \ldots, N$. Let us denote by $\mathbf{Q} = \mathbf{R}_L = \mathrm{E}\{\mathbf{x}_L \mathbf{x}_L^T\}$ the correlation matrix of the ranked ordered samples. Similarly let $\mathbf{R}_{L\ell} = \mathrm{E}\{\mathbf{x}_{L\ell}\mathbf{x}^T_{L\ell}\}$ define the correlation matrix of time and ranked ordered samples. By definition both $\mathbf{R}_{L\ell}$ and $\mathbf{R}_L$ are positive semi-definite. Following similar arguments to [13, pp. 190-191], we argue that the aforementioned matrices are positive definite if the random variables of concern are linearly independent. It is trivial to show that the sum of the eigenvalues is the same in both $\mathbf{R}_{L\ell}$ and $\mathbf{Q}$, that is

$$\sum_{i=1}^{N^2} \lambda_i(\mathbf{R}_{L\ell}) = \sum_{i=1}^{N} \lambda_i(\mathbf{Q}) = N(\sigma^2 + s^2) \quad (17)$$

where $\lambda_i(\mathbf{Q})$ is the i-th eigenvalue of $\mathbf{Q}$ and σ^2 is the noise variance. Let $\mathbf{0}$ denote an $N \times N$ matrix of zeroes. By employing elementary similarity transformations it can be shown that $\mathbf{R}_{L\ell}$ is similar to a matrix having the following structure:

$$\begin{bmatrix} \mathbf{G} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{22} & \dots & \mathbf{A}_{2N} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{A}_{N2} & \dots & \mathbf{A}_{NN} \end{bmatrix} \quad (18)$$

where

$$\mathbf{G} = \begin{bmatrix} \frac{1}{N}Q_{11} & \frac{1}{N(N-1)}Q_{12} & \dots & \frac{1}{N(N-1)}Q_{1N} \\ \frac{1}{N(N-1)}Q_{12} & \frac{1}{N}Q_{22} & \dots & \frac{1}{N(N-1)}Q_{2N} \\ \vdots & & \ddots & \vdots \\ \frac{1}{N(N-1)}Q_{1N} & \frac{1}{N(N-1)}Q_{2N} & \dots & \frac{1}{N}Q_{NN} \end{bmatrix} \quad (19)$$

and $\mathbf{A}_{ij}, i, j = 2, \dots, N$ are appropriate matrices. The eigenvalues of matrix $\mathbf{G}$ can be obtained analytically, i.e.

$$\lambda_i(\mathbf{G}) = \frac{1}{N}\lambda_i(\mathbf{Q}). \quad (20)$$

Accordingly, we have found that under the i.i.d. assumption N out of the N^2 eigenvalues of $\mathbf{R}_{L\ell}$ can be obtained from the eigenvalues of $\mathbf{R}_L$ divided by N. The eigenvalue distribution of the correlation matrix of ranked ordered samples $\mathbf{R}_L$ and the time and ranked ordered samples $\mathbf{R}_{L\ell}$ is plotted in Figure 1 for uniform, Gaussian and Laplacian noise distribution having zero mean and unit variance, when $s = 5$ and N=5. In the following subsection, we demonstrate that it is also possible to derive upper and lower bounds for the smallest and the largest eigenvalue of $\mathbf{R}_{L\ell}$.

4.1. Upper and lower bounds on the extreme eigenvalues of the correlation matrix of time and ranked ordered samples

The correlation matrix of the time and ranked ordered samples can be decomposed as

$$\mathbf{R}_{L\ell} = \frac{1}{N}\text{diag}\left(Q_{11}, Q_{22}, \dots, Q_{NN}\right) \otimes \mathbf{I} + \mathbf{B} \quad (21)$$

where diag() denotes the diagonal matrix whose diagonal elements are those inside parentheses, $\mathbf{I}$ is the $N \times N$ identity matrix, $\otimes$ denotes the Kronecker product and $\mathbf{B} = \mathbf{\Theta} \otimes \mathbf{W}$. The $N \times N$ matrices $\mathbf{\Theta}$ and $\mathbf{W}$ are given by

$$\mathbf{\Theta} = \begin{bmatrix} 0 & Q_{12} & \dots & Q_{1N} \\ Q_{12} & 0 & \dots & Q_{2N} \\ \vdots & & \ddots & \vdots \\ Q_{1N} & Q_{2N} & \dots & 0 \end{bmatrix} \quad (22)$$

$$\mathbf{W} = \frac{1}{N(N-1)}\left(\mathbf{1}\mathbf{1}^T - \mathbf{I}\right) \quad (23)$$

where $\mathbf{1}$ is the $N \times 1$ vector of ones.

$\mathbf{B}$ has N^2 eigenvalues whose sum equals zero. The sum of the N eigenvalues of both $\mathbf{\Theta}$ and $\mathbf{W}$ equals zero as well. Therefore, the smallest eigenvalue of matrix $\mathbf{\Theta}$ is negative while its largest eigenvalue is positive. It can be shown that $\mathbf{W}$ has two distinct eigenvalues, i.e., $\lambda_1(\mathbf{W}) = \frac{1}{N}$ with multiplicity 1 and $\lambda_2(\mathbf{W}) = -\frac{1}{N(N-1)}$ with multiplicity $(N-1)$. Accordingly, the eigenvalues of $\mathbf{B}$ are as follows:

$$-\frac{\lambda_i(\mathbf{\Theta})}{N(N-1),} \text{ with multiplicity } N-1,\ i = 1, 2, \dots, N$$
$$\lambda_i(\mathbf{\Theta})\frac{1}{N}, \quad i = 1, 2, \dots, N \quad (24)$$

where $\lambda_i(\mathbf{\Theta})$ are the eigenvalues of matrix $\mathbf{\Theta}$. The previous discussion yields the following expressions:

$$\lambda_{\max}(\mathbf{B}) = \max\left(-\frac{\lambda_{\min}(\mathbf{\Theta})}{N(N-1)}, \frac{\lambda_{\max}(\mathbf{\Theta})}{N}\right)$$
$$\lambda_{\min}(\mathbf{B}) = \min\left(-\frac{\lambda_{\max}(\mathbf{\Theta})}{N(N-1)}, \frac{\lambda_{\min}(\mathbf{\Theta})}{N}\right). \quad (25)$$

Accordingly, we need to relate the eigenvalues of matrices $\mathbf{B}$ and $\mathbf{\Theta}$ to the eigenvalues of the correlation matrix of ranked ordered samples $\mathbf{R}_L = \mathbf{Q}$. The latter matrix can be decomposed as follows:

$$\mathbf{Q} = \mathbf{\Theta} + \text{diag}\left(Q_{11}, Q_{22}, \dots, Q_{NN}\right). \quad (26)$$

Let us define by $\zeta_{\max}$ and $\zeta_{\min}$ the maximum and minimum diagonal element of the correlation matrix of the ranked ordered samples, respectively, i.e.:

$$\zeta_{\max} = \max\left(Q_{11}, Q_{22}, \dots, Q_{NN}\right) \quad (27)$$
$$\zeta_{\min} = \min\left(Q_{11}, Q_{22}, \dots, Q_{NN}\right). \quad (28)$$

By applying Theorem 8.1.5 [14, p. 396] we obtain

$$\alpha \leq \lambda_{\max}(\mathbf{\Theta}) \leq \beta$$
$$\gamma \leq \lambda_{\min}(\mathbf{\Theta}) \leq \delta \quad (29)$$

where

$$\alpha = \lambda_{\max}(\mathbf{Q}) - \zeta_{\max} \qquad \beta = \lambda_{\max}(\mathbf{Q}) - \zeta_{\min}$$
$$\gamma = \lambda_{\min}(\mathbf{Q}) - \zeta_{\max} \qquad \delta = \lambda_{\min}(\mathbf{Q}) - \zeta_{\min}. \quad (30)$$

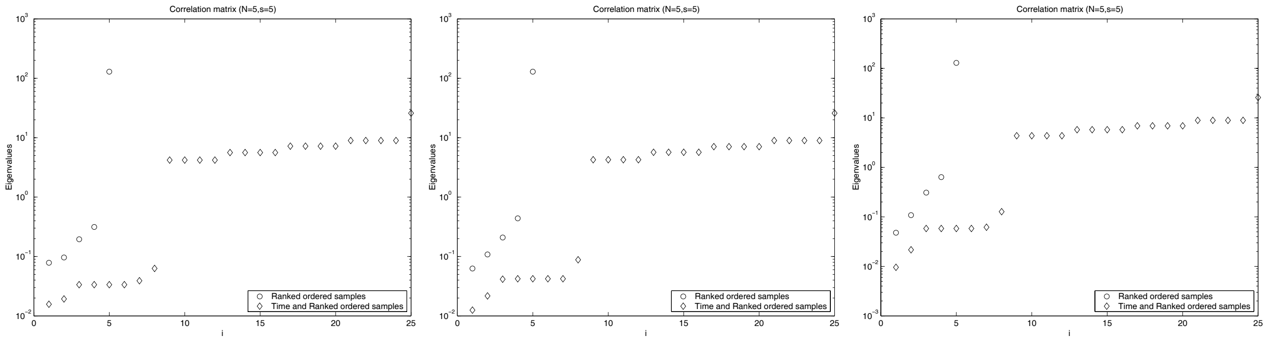
The application of the same theorem yields also

$$\frac{\zeta_{\max}}{N} + \lambda_{\min}(\mathbf{B}) \leq \lambda_{\max}(\mathbf{R}_{L\ell}) \leq \frac{\zeta_{\max}}{N} + \lambda_{\max}(\mathbf{B})$$
$$\frac{\zeta_{\min}}{N} + \lambda_{\min}(\mathbf{B}) \leq \lambda_{\min}(\mathbf{R}_{L\ell}) \leq \frac{\zeta_{\min}}{N} + \lambda_{\max}(\mathbf{B}) \quad (31)$$

By combining (25) and the inequalities (29), (30) and (31) we get

$$\frac{\lambda_{\min}(\mathbf{R}_L)}{N} \leq \lambda_{\max}(\mathbf{R}_{L\ell}) \leq \frac{\lambda_{\max}(\mathbf{R}_L) + \zeta_{\max} - \zeta_{\min}}{N}$$
$$\frac{\lambda_{\min}(\mathbf{R}_L) - (\zeta_{\max} - \zeta_{\min})}{N} \leq \lambda_{\min}(\mathbf{R}_{L\ell}) \leq \frac{\lambda_{\max}(\mathbf{R}_L)}{N}. \quad (32)$$

The numerical computations summarized in Table 1 indicate that the extreme eigenvalues of $\mathbf{R}_{L\ell}$ are equal to the extreme eigenvalues of $\mathbf{R}_L$ divided by N, a result that is within the intervals predicted by the analysis described above. As a consequence the eigenvalue spread of the correlation matrix of time and ranked ordered samples is the same with that of the correlation matrix of ranked ordered samples.



(a) (b) (c)

Fig. 1. Eigenvalue distribution of ranked ordered samples $\mathbf{R}_L$ and the time and ranked ordered samples $\mathbf{R}_{L\ell}$ when $s = 5$ and $N = 5$ for (a) uniform, (b) Gaussian, and (c) Laplacian noise distribution having zero mean and unit variance.

Table 1. Smallest and largest eigenvalues of $\mathbf{R}_L$ and $\mathbf{R}_{L\ell}$ for uniform, Gaussian and Laplacian parent distribution having mean $s = 5.0$ and unit variance.

parent distribution	N	$\lambda_{\min}(\mathbf{R}_L)$	$\lambda_{\min}(\mathbf{R}_{L\ell})$	$\lambda_{\max}(\mathbf{R}_L)$	$\lambda_{\max}(\mathbf{R}_{L\ell})$
uniform	5	0.078672	0.015734	129.314823	25.862965
	7	0.043776	0.006254	181.228691	25.889813
	9	0.028086	0.003121	233.176925	25.908547
Gaussian	5	0.062604	0.012521	129.180824	25.836165
	7	0.031849	0.004550	181.037213	25.862459
	9	0.019249	0.002139	232.937543	25.881949
Laplacian	5	0.047948	0.009590	128.898323	25.779665
	7	0.020916	0.002988	180.642761	25.806109
	9	0.007333	0.000815	232.457473	25.828608

5. EIGENVALUES OF THE CORRELATION MATRIX OF RANKED ORDERED SAMPLES

In the preceding analysis we have employed the correlation matrix of the ranked ordered (noisy) observations. Under the assumption of a constant signal corrupted by additive white noise, the latter correlation matrix can be expressed in terms of the correlation matrix of ranked ordered noise samples, $\mathbf{\Phi} = \mathrm{E}\{\mathbf{v}_L \mathbf{v}_L^T\}$, as follows:

$$\mathbf{R}_L = \mathbf{\Phi} + s\left(\mathbf{1}\boldsymbol{\mu}^T + \boldsymbol{\mu}\mathbf{1}^T\right) + s^2\mathbf{1}\mathbf{1}^T \tag{33}$$

where $\boldsymbol{\mu} = \mathrm{E}\{\mathbf{v}_L\}$ is the vector of the expected values of the order statistics of noise samples. Subsequently, we analyze the behavior of the eigenvalues of $\mathbf{R}_L$. For symmetric noise distributions about zero, it can be shown that the matrix $\mathbf{H} = \left(\mathbf{1}\boldsymbol{\mu}^T + \boldsymbol{\mu}\mathbf{1}^T\right)$ is similar to [15]

$$\mathbf{H}' = \begin{bmatrix} \mathbf{0} & \mathbf{\Gamma} - \mathbf{\Upsilon}\mathbf{\Delta} \\ \mathbf{\Gamma} + \mathbf{\Upsilon}\mathbf{\Delta} & \mathbf{0} \end{bmatrix}. \tag{34}$$

For N odd, the matrices $\mathbf{\Gamma}$ and $\mathbf{\Delta}$ in (34) are given, respectively, by:

$$\mathbf{\Gamma} = \tilde{\boldsymbol{\mu}}\tilde{\mathbf{1}}^T + \tilde{\mathbf{1}}\tilde{\boldsymbol{\mu}}^T \tag{35}$$
$$\mathbf{\Delta} = \tilde{\mathbf{1}}\tilde{\boldsymbol{\mu}}^T - \mathbf{\Upsilon}\tilde{\boldsymbol{\mu}}\tilde{\mathbf{1}}^T \tag{36}$$

where $\tilde{\boldsymbol{\mu}}$ is the vector of the expected values of $v_{(1)}, v_{(2)}, \ldots, v_{(\frac{N-1}{2})}$, $\tilde{\mathbf{1}}$ is the $(\frac{N-1}{2})$-dimensional vector of ones and $\mathbf{\Upsilon}$ is the $(\frac{N-1}{2} \times \frac{N-1}{2})$ matrix that has ones along the secondary diagonal and zeros elsewhere. It can be further proved that the matrix defined in (34) has $(N-2)$ zero eigenvalues and the remaining two non-zero ones are given by

$$\lambda_{1,2}(\mathbf{H}) = \lambda_{1,2}(\mathbf{H}') = \pm\sqrt{2N\tilde{\boldsymbol{\mu}}^T\tilde{\boldsymbol{\mu}}}. \tag{37}$$

Without any loss of generality, if $s > 0$, we obtain

$$\lambda_{1,2}(s\,\mathbf{H}) = \pm\eta = \pm s\sqrt{2N\tilde{\boldsymbol{\mu}}^T\tilde{\boldsymbol{\mu}}}. \tag{38}$$

By applying Theorem 8.1.5 [14, pp. 396] it can be shown that

$$\lambda_{\min}(\mathbf{\Phi} + s\,\mathbf{H}) \leq \lambda_{\min}(\mathbf{\Phi}) + \eta \tag{39}$$
$$\lambda_{\max}(\mathbf{\Phi} + s\,\mathbf{H}) \leq \lambda_{\max}(\mathbf{\Phi}) + \eta. \tag{40}$$

Theorem 8.1.8 [14, pp. 397] asserts that there exist nonnegative coefficients $k_1, k_2, \ldots, k_N$ such that

$$\lambda_{(i)}(\mathbf{R}_L) = \lambda_{(i)}(\mathbf{\Phi} + s\,\mathbf{H}) + k_i s^2 N \tag{41}$$

where the eigenvalues are arranged in ascending order of magnitude.

If

$$\lambda_{(i)}(\mathbf{R}_L) = \lambda_{(i+1)}(\mathbf{\Phi} + s\,\mathbf{H}) = \lambda_{(i)}(\mathbf{\Phi}), \;\; i = 1, 2, \ldots, N-2 \tag{42}$$

then it can be shown that

$$\lambda_{\max}(\mathbf{R}_L) \le Ns^2 + \lambda_{(N-1)}(\mathbf{\Phi}) + \lambda_{\max}(\mathbf{\Phi}) \tag{43}$$

The assumption (42) has been verified in numerical computations, as can be seen in Table 2. In the computations we have used the tables from [16]. The validity of the upper bound in (43) is demonstrated in Table 3 for uniform, Gaussian and Laplacian parent distributions having mean $s = 5.0$ and unit variance.

Table 2. Eigenvalues of matrices $\mathbf{\Phi}$, $\mathbf{\Phi}+s\mathbf{H}$, and $\mathbf{R}_L$ for Gaussian noise distribution having zero mean and unit variance, when $s = 5.0$.

N	i	$\lambda_{(i)}(\mathbf{\Phi})$	$\lambda_{(i)}(\mathbf{\Phi}+s\mathbf{H})$	$\lambda_{(i)}(\mathbf{R}_L)$
5	1	0.062604	-17.716818	0.062604
	2	0.108597	0.062604	0.108505
	3	0.207441	0.108607	0.207441
	4	1.000000	0.207441	0.440627
	5	3.621358	22.338166	129.180824
7	1	0.031849	-26.573335	0.031849
	2	0.046911	0.031849	0.046911
	3	0.073496	0.046911	0.073496
	4	0.124484	0.073496	0.124328
	5	0.227863	0.124495	0.227863
	6	1.000000	0.227863	0.458341
	7	5.495397	33.068721	181.037213
9	1	0.019249	-35.481640	0.019249
	2	0.025896	0.019249	0.025896
	3	0.036136	0.025896	0.036136
	4	0.052806	0.036136	0.052804
	5	0.081777	0.052806	0.081777
	6	0.135986	0.081777	0.135798
	7	0.241604	0.135996	0.241604
	8	1.000000	0.241604	0.469194
	9	7.406547	43.888177	232.937543

Accordingly, the smallest eigenvalue of $\mathbf{R}_L$ and consequently $\mathbf{R}_{L\ell}$ is controlled exclusively by the noise statistics, that is, the smallest eigenvalue of the correlation matrix of the ordered noise samples. On the contrary, the largest eigenvalue of both $\mathbf{R}_L$ and $\mathbf{R}_{L\ell}$ is influenced by the dimensionality of the observation window, the true constant signal value, and the noise variance.

Table 3. Validity of the upper bound in (43) for uniform, Gaussian and Laplacian parent distributions having when $s = 5.0$ and unit variance.

parent distribution	N	$\lambda_{\max}(\mathbf{R}_L)$	Upper bound
uniform	5	129.314823	129.667237
	7	181.228691	181.595860
	9	233.176925	233.555530
Gaussian	5	129.180824	129.621358
	7	181.037213	181.495397
	9	232.937543	233.406547
Laplacian	5	128.898323	129.608924
	7	180.642761	181.438250
	9	232.457473	233.299820

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