Adaptive LMS L-Filters for Smoothing Noisy Images

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Abstract

Several adaptive LMS L-filters, both constrained and unconstrained ones, are developed for noise suppression in images and being compared in this paper. First, the location-invariant LMS L-filter for a nonconstant signal corrupted by zero-mean additive white noise is derived. Subsequently, the normalized and the sign LMS L-filters are studied. It is shown that both these filters turn to be identical for a certain choice of the adaptation step-size. A modified LMS L-filter with nonhomogeneous step-sizes is also proposed in order to accelerate the rate of convergence of the adaptive L-filter. Finally, a signal-dependent adaptive filter structure is developed to allow a separate treatment of the pixels that are close to the edges from the pixels that belong to homogeneous image regions.

1 Introduction

A multitude of nonlinear techniques has been proven a successful alternative to the linear techniques in adaptive signal processing. For a review of the nonlinear filter classes the reader may consult [1]. One of the best known families is based on the order statistics. It uses the concept of sample ordering. There is now a multitude of nonlinear filters based on data ordering. Among them are the L-filters whose output is defined as a linear combination of the order statistics of the input sequence. Recently, the adaptation of the coefficients employed in order statistic filters by using linear adaptive signal processing techniques has received much attention in the literature [2, 3, 4, 5]. In this paper, we shall confine ourselves to the design of adaptive L-filters.

The main contribution of this paper is in the design and comparison of several adaptive L-filters for noise suppression in images. The properties of the developed adaptive L-filters are studied as well. Another primary goal is to establish links between the adaptive L-filters under study and other algorithms developed elsewhere.

2 Location-invariant LMS L-filter

In this section, the location-invariant LMS L-filter for a nonconstant signal corrupted by zero-mean additive white noise is derived. Let us consider that the observed signal $x(k)$ can be expressed as a sum of an arbitrary signal $s(k)$ plus zero-mean additive white noise, i.e., $x(k) = s(k) + n(k)$. Let $\mathbf{x}_r(k)$ be the ordered tap-input vector at time instant k given by

$$\mathbf{x}_r(k) = \left(x_{(1)}(k), x_{(2)}(k), \ldots, x_{(N)}(k)\right)^T \tag{1}$$

where $x_{(i)}(k)$ denotes the i-th largest observation in the $N \times 1$ tap-input vector

$$\mathbf{x}(k) = \left(x(k - \frac{N-1}{2}), \ldots, x(k), \ldots, x(k + \frac{N-1}{2})\right)^T \tag{2}$$

The length of the tap-input vector N is assumed to be odd. We are seeking the L-filter whose output at time instant k

$$y(k) = \mathbf{a}^T(k)\, \mathbf{x}_r(k) \tag{3}$$

minimizes the MSE $J(k) = \mathrm{E}\left[(y(k) - s(k))^2\right]$ between the filter output $y(k)$ and the desired response $s(k)$ subject to the constraint

$$\mathbf{1}_N^T\, \mathbf{a}(k) = 1 \tag{4}$$

where $\mathbf{1}_N$ is the $N \times 1$ unitary vector, i.e., $\mathbf{1}_N = (1, 1, \ldots, 1)^T$. The constraint (4) ensures that the filter preserves the zero-frequency or dc signals. Let $\nu = (N+1)/2$. By employing (4), we can partition the L-filter coefficient vector as follows

$$\mathbf{a}(k) = \left(\mathbf{a}_1^T(k) | a_\nu(k) | \mathbf{a}_2^T(k)\right)^T \tag{5}$$

where $\mathbf{a}_1(k)$, $\mathbf{a}_2(k)$ are $(N-1)/2 \times 1$ vectors given by

$$\mathbf{a}_1(k) = (a_1(k), \ldots, a_{\nu-1}(k))^T \quad \mathbf{a}_2(k) = (a_{\nu+1}(k), \ldots, a_N(k))^T \tag{6}$$

and the coefficient for the median input sample is evaluated as follows:

$$a_\nu(k) = 1 - \mathbf{1}_{\nu-1}^T \mathbf{a}_1(k) - \mathbf{1}_{\nu-1}^T \mathbf{a}_2(k). \tag{7}$$

Let $\mathbf{a}'(k)$ be the reduced L-filter coefficient vector

$$\mathbf{a}'(k) = \left(\mathbf{a}_1^T(k) | \mathbf{a}_2^T(k)\right)^T \tag{8}$$

and $\hat{\mathbf{x}}_r(k)$ be the following vector

$$\hat{\mathbf{x}}_r(k) = \begin{bmatrix} \mathbf{x}_{r1}(k) - x_{(\nu)}(k)\mathbf{1} \\ \mathbf{x}_{r2}(k) - x_{(\nu)}(k)\mathbf{1} \end{bmatrix} \quad (9)$$

Following the analysis in [4] it can be proven that the LMS recursive relation for updating the filter coefficients is

$$\hat{\mathbf{a}}'(k+1) = \hat{\mathbf{a}}'(k) + \mu\, \varepsilon(k)\, \hat{\mathbf{x}}_r(k) \quad (10)$$

where $\varepsilon(k)$ is the estimation error at time instant k, i.e., $\varepsilon(k) = s(k) - y(k)$. Eq. (10) along with (7) constitute the *location-invariant LMS L-filter.*

3 Variants of unconstrained LMS L-filters

In the sequel, we deal with the unconstrained LMS adaptive L-filter [3] whose coefficients are updated by using the following recursive formula:

$$\hat{\mathbf{a}}(k+1) = \hat{\mathbf{a}}(k) + \mu(k)\, \varepsilon(k)\, \mathbf{x}_r(k). \quad (11)$$

Three modifications of the unconstrained LMS adaptive L-filter are discussed, namely, the normalized LMS L-filter, the sign LMS L-filter and the modified LMS L-filter with nonhomogeneous step-sizes.

The normalized LMS L-filter does not share the difficulty in choosing the appropriate step-size parameter that is frequently met in the ordinary LMS L-filter (e.g. when $\mu(k) = \mu$). The adaptation of the normalized LMS L-filter coefficients is described by

$$\hat{\mathbf{a}}(k+1) = \hat{\mathbf{a}}(k) + \frac{\mu_0}{||\mathbf{x}_r(k)||^2}\, \varepsilon(k)\, \mathbf{x}_r(k). \quad (12)$$

The recursive equation (12) is equivalent to the linear normalized LMS algorithm. It can easily be shown that μ_0 should be chosen to satisfy the inequality $0 < \mu_0 \leq \frac{2}{3}$.

The derivation of the majority of adaptive filter algorithms relies on the minimization of MSE criterion. Another optimization criterion that is encountered in image processing is the Mean Absolute Error (MAE) criterion. The so called *sign LMS L-filter* that minimizes the MAE between the filter output and the desired response is derived as well. The adaptation of the sign LMS L-filter coefficients is given by

$$\mathbf{a}(k+1) = \mathbf{a}(k) + \mu\, \mathrm{sgn}\, [\varepsilon(k)]\; \mathbf{x}_r(k) \quad (13)$$

where $\mathrm{sgn}\,[\cdot]$ denotes the sign of the bracketed expression:

$$\mathrm{sgn}\,[x] = \begin{cases} 1 & \text{if } x > 0 \\ \\ -1 & \text{if } x < 0. \end{cases} \quad (14)$$

It is worth noting that by substituting the following step-size sequence

$$\mu(k) = \frac{\mu_0 |\varepsilon(k)|}{\mathbf{x}_r^T(k)\mathbf{x}_r(k)} \qquad 0 < \mu_0 < 1 \quad (15)$$

into (13), the updating formula for the normalized LMS L-filter coefficients is obtained. Therefore, the normalized LMS L-filter (12) and the sign LMS L-filter (13) for the choice of the step-size sequence (15) are identical.

Subsequently, a modified LMS L-filter with nonhomogeneous step-sizes is introduced in order to accelerate the rate of convergence of the adaptive L-filter by allowing the convergence of each L-filter coefficient to be controlled by a separate step-size parameter. We have found by experiments that the following step-size sequence

$$\mu_i(k) = \mu_0 \frac{\sum_{j=0}^{k} x_{(i)}(k-j)}{\sum_{j=0}^{k} x_{(1)}(k-j)} \quad (16)$$

gives results comparable to those obtained by using the normalized LMS L-filter algorithm. By using (16), the modified LMS L-filter updating formula is written as follows:

$$\mathbf{a}(k+1) = \mathbf{a}(k) + \varepsilon(k)\, \mathbf{M}(k)\mathbf{x}_r(k) \quad (17)$$

with $\mathbf{M}(k) = \mathrm{diag}\,[\mu_1(k), \mu_2(k), \ldots, \mu_N(k)]$ denoting a diagonal matrix.

Finally, a signal-dependent adaptive filter structure is developed. It aims at a different treatment of the image pixels close to the edges from the pixels that belong to homogeneous regions. The signal-dependent adaptive L-filter structure consists of two LMS adaptive L-filters whose outputs $y_L(k)$ and $y_H(k)$ are combined to give the final response as follows:

$$\begin{aligned} y(k) &= y_L(k) + \beta(k)\{y_H(k) - y_L(k)\} = \\ &= \beta(k) y_H(k) + [1 - \beta(k)]\, y_L(k) \end{aligned} \quad (18)$$

where $\beta(k)$ is a local measure of signal activity that varies between 0 and 1. The local Signal-to-Noise Ratio (SNR) measure derived in [6] has been employed, i.e.

$$\beta(k) = 1 - \frac{\sigma_n^2}{\hat{\sigma}_x^2(k)} \quad (19)$$

where σ_n^2 is the noise variance and $\hat{\sigma}_x^2(k)$ is the local variance of the noisy input observations.

4 Simulation examples

Only one set of experiments conducted on images is described due to lack of space. In this set, we presuppose that a reference image (e.g. the original image) is available. In practice, reference images are usually transmitted through TV telecommunication channels to measure the performance of the channel. In such cases, the proposed adaptive L-filters are very useful, if the design of an optimal filter for the specific channel characteristics is required. Our goal is to compare the performance of the adaptive L-filters under study. Two criteria have been employed,

Table 1: Noise reduction and Mean Absolute Error reduction (in dB) achieved by the various LMS adaptive L-filters in the restoration of “Lenna” corrupted by mixed impulsive and additive Gaussian noise.

Method	NR	MAER
median 3×3	-8.756	-8.147
location-invariant LMS L-filter 3×3 ($\mu = 5 \times 10^{-7}$)	-9.747	-9.192
modified LMS L-filter 3×3 with nonhomogeneous step-sizes ($\mu_0 = 5 \times 10^{-7}$)	-11.216	-10.867
normalized LMS L-filter 3×3 ($\mu_0 = 0.8$)	-11.281	-11.071
signal-dependent normalized LMS L-filter structure (equal dimensions 3×3; $\beta_t = 0.75$)	-9.024	-9.552
signal-dependent normalized LMS L-filter structure (L: 5×5, H: 3×3; $\beta_t = 0.75$)	-13.224	-13.928

namely, the noise reduction index (NR) defined as the ratio of the output noise power to the input noise power, i.e.,

$$\mathrm{NR} = 10 \log \frac{\frac{1}{K\,L} \sum_{i=1}^{K} \sum_{j=1}^{L} (y(i,j) - s(i,j))^2}{\frac{1}{K\,L} \sum_{i=1}^{K} \sum_{j=1}^{L} (x(i,j) - s(i,j))^2} \quad \text{(in dB)} \tag{20}$$

and the Mean Absolute Error Reduction (MAER) defined as the ratio of the mean absolute error in the output to the mean absolute error in the input, i.e.,

$$\mathrm{MAER} = 20 \log \frac{\frac{1}{K\,L} \sum_{i=1}^{K} \sum_{j=1}^{L} |y(i,j) - s(i,j)|}{\frac{1}{K\,L} \sum_{i=1}^{K} \sum_{j=1}^{L} |x(i,j) - s(i,j)|} \quad \text{(in dB)} \tag{21}$$

In (20) and (21), $s(i,j)$ is the original image pixel, $x(i,j)$ denotes the same image pixel corrupted by noise and $y(i,j)$ is the filter output at the same image pixel. K, L are the number of image rows and columns respectively.

The NR as well as the MAER achieved by the location-invariant LMS L-filter, the normalized LMS L-filter and the modified LMS L-filter with nonhomogeneous step-sizes, all of dimensions 3×3 in the case of mixed impulsive and additive Gaussian noise are listed in Table 1. In the same table, we have also included the corresponding figures of merit for the 3×3 median filter.

As can be seen, the condition for location-invariant estimation is strict enough and the resulting adaptive L-filter is only 1 dB better than the median filter with respect to both quantitative measures. The modified LMS L-filter with nonhomogeneous step-sizes is the second best adaptive L-filter yielding an almost 2.5 dB better NR and MAER compared to the median filter. The normalized LMS L-filter achieves the best performance both in the MSE sense as well as in MAE sense. Especially for the MAE criterion, this performance was expected, since we have already demonstrated the connection between the sign LMS L-filter and the normalized one. The optimal value of parameter μ_0 has been found experimentally. In addition, we have tested the performance of two signal-dependent adaptive L-filter structures. The first one uses two 3×3 adaptive L-filters that are trained by different regions of the corrupted input image. More specifically, the pixels that belong to the homogeneous image regions are used to adapt the coefficients of the one adaptive L-filter, while those that are close to the image edges are used to adapt the coefficients of the second adaptive L-filter. Any of the adaptive L-filters that have been described in this paper (e.g. the location-invariant LMS L-filter, the normalized LMS L-filter or the modified LMS L-filter with nonhomogeneous step-sizes) can be included in the signal-dependent structure. In the experiments described, we have used the normalized LMS L-filter. By inspecting Table 1, it is seen that the signal-dependent adaptive L-filter structure provides the best results, when the window of the normalized LMS L-filter that is used in homogeneous image regions is of larger dimensions (e.g. 5×5) than that of the adaptive LMS L-filter that is trained by pixels close to the image edges. The observed superior performance is due to the larger window size that is used to filter the noise in homogeneous regions.

References

[1] I. Pitas, and A.N. Venetsanopoulos, *Nonlinear Digital Filters: Principles and Applications*. Dordrecht, Holland: Kluwer Academic, 1990.

[2] C. Kotropoulos, and I. Pitas, “Adaptive nonlinear filters for digital signal/image processing,” in *2-D and 3-D Signal Processing Techniques and Applications* (C.T. Leondes, ed.), Academic Press, in press.

[3] I. Pitas, and A.N. Venetsanopoulos, “Adaptive filters based on order statistics,” *IEEE Trans. on Signal Processing*, vol. 39, no. 2, pp. 518–522, February 1991.

[4] C. Kotropoulos, and I. Pitas, “Constrained adaptive LMS L-filters,” *Signal Processing*, vol. 26, no. 3, pp. 335–358, 1992.

[5] G.A. Williamson, and P.M. Clarkson, “On signal recovery with adaptive order statistic filters, ” *IEEE Trans. on Signal Processing*, vol. SP-40, no. 10, pp. 2622–2626, October 1992.

[6] R. Bernstein, “Adaptive nonlinear filters for simultaneous removal of different kinds of noise in images,” *IEEE Trans. on Circuits and Systems*, vol. CAS-34, no. 11, pp. 1275–1291, November 1987.

[7] C. Kotropoulos, and I. Pitas, “Adaptive LMS L-filters for noise suppression in images,” *IEEE Trans. on Image Processing*, under review.