VORONOI TESSELLATION AND DELAUNEY TRIANGULATION USING EUCLIDEAN DISK GROWING IN $\mathcal{Z}^2$

C. Kotropoulos I. Pitas A. Maglara

Department of Electrical Engineering
University of Thessaloniki
Thessaloniki 540 06, GREECE

ABSTRACT

A new approximation of Euclidean distance in $\mathcal{Z}^2$ is proposed and a novel algorithm for the computation of Voronoi tessellation and Delauney triangulation is presented based on the above-mentioned approximation of Euclidean distance. The proposed method has low computational complexity (of order $\mathcal{O}(1/N)$) and allows parallel implementation. Mathematical Morphology is used to implement the Voronoi tessellation and the Delauney triangulation.

1. INTRODUCTION

Voronoi tessellation is a very important tool in computational geometry [3], object recognition [2] and image analysis [4]. Several important problems can be solved by employing Voronoi tessellation, for example Delauney triangulation, convex hull, object decomposition into simple components (triangles). The computation of Voronoi tessellation is closely related to distance transformations as well as to the computation of skeletons by influence zones. All the above-mentioned research areas rely on the availability of approximations of the Euclidean distance function in a discrete grid that are as accurate as possible. The city-block, the chessboard and the octagonal distance functions have been proposed as approximations to the Euclidean distance function in [6]. Efficient sequential algorithms for generating Euclidean distance maps are discussed in [8,14]. Several families of distance transformations are generalized to higher dimensions and are compared to computed distances with Euclidean distance in [9,13]. The local distances used in 3×3, 5×5 and 7×7 neighborhoods have been optimized by minimizing a multitude of criteria in [11,12,18]. An efficient version of the uniform cost algorithm is applied to general distance transformations in [16]. The decomposition of Euclidean distance structuring element by employing a set of 3×3 grayscale morphological erosions with suitable weighted structuring elements and combining the erosion outputs using a minimum operator is presented in [20]. Several approaches to the computation of skeletons by influence zones can be found in [7,15,17,19,21].

The research reported in this paper has been supported by ESPRIT Basic Research Action project NAT (7130). C. Kotropoulos has also been supported by a scholarship from the State Scholarship Foundation of Greece and the Bodosaki Foundation

A new approximation of Euclidean distance in $\mathcal{Z}^2$ is proposed and a novel algorithm for the computation of Voronoi tessellation and Delauney triangulation is presented based on the above-mentioned approximation of Euclidean distance. Mathematical Morphology [1,5] is used to implement the Voronoi tessellation and the Delauney triangulation.

Let $X = \{x_1, x_2, \ldots, x_N\}$ be a set of N points defined on $W \subseteq \mathcal{R}^N$ or $W \subseteq \mathcal{Z}^N$ and $d(.)$ be a distance function. The set of all points $x \in W$ that are closer to a given point $x_i \in X$ than any other point $x_j \in X$ $j \neq i$ is called *Voronoi region* of x_i. It is defined as follows:

$$V(i) = \{x \in W : d(x, x_i) < d(x, x_j) \;\; j \neq i\} \tag{1}$$

The union of Voronoi regions for all $x_i \in X$ is called *Voronoi diagram* of X, i.e.:

$$\mathrm{Vor}(X) = \bigcup_{i=1}^{N} V(i) \tag{2}$$

In the case of $\mathcal{Z}^2$, the definitions given above imply that the construction of the Voronoi diagram in $\mathcal{Z}^2$ can be easily implemented, if we are able to find a simple and efficient growth mechanism for all points in X. At each step m of the growing procedure, the points in $\mathcal{Z}^2$ having smaller distance from x_i than from any other x_j, $j \neq i$ should be appended to each $x_i \in X$. It is well known that Euclidean distance can be approximated in $\mathcal{Z}^2$ by using non Euclidean metrics described by employing the morphological operator *dilation* [6,9,11,20,21]. Therefore, we shall rely on the dilation operator. Its definition follows [1,5]:

$$\begin{aligned} Y \oplus B^s &= \bigcup_{b \in B} Y_{-b} = \{x \in \mathcal{Z}^2 : B_x \cap Y \neq \emptyset\} \\ &= \{x \in \mathcal{Z}^2 : B_x \uparrow Y\} \end{aligned} \tag{3}$$

where B^s is the symmetric of the structuring element B with respect to the origin, Y_b is a translate version of the set Y centered at b. The symbol $\uparrow$ states that the set B_x hits Y [1].

The outline of the paper follows. The proposed approximation of Euclidean distance in $\mathcal{Z}^2$ is described in Section 2. The computation of Voronoi tessellation based on the above-mentioned approximation of Euclidean distance is treated in Section 3. An application of the proposed implementation of Voronoi tessellation to the derivation of Delauney triangulation of a polygonal object is presented in Section 4.

0-7803-0946-4/93 $3.00

2. EUCLIDEAN DISK GROWING IN $\mathcal{Z}^2$

The most popular non Euclidean metrics that are used to approximate Euclidean distance in $\mathcal{Z}^2$ are the city-block, the chessboard and the octagonal distance fuctions [6]. Another class of more accurate distance functions is the so-called chamfer distances proposed in [9,11].

Let $d_e(x,y)$ denote the Euclidean distance between two points $x=(x_1,x_2)$ and $y=(y_1,y_2)$ in $\mathcal{R}^2$, i.e.:

$$d_e(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2} \quad x,y\in\mathcal{R}^2 \qquad (4)$$

The following distance function between two points in $\mathcal{Z}^2$ has been proposed in an attempt to translate (4) to a formula suitable for $\mathcal{Z}^2$[6]:

$$d_z(x,y)=\inf\{k\in\mathcal{N}: k-0.5\le d_e(x,y)<k+0.5\} \qquad (5)$$

where $\mathcal{N}$ denotes the set of natural numbers. It is worth noting that (5) is not a metric, unless the triangle inequality is relaxed. In this paper, a class of more accurate distance functions than the uniform-step-distance (USD) and the periodically-uniform-step distance (PUSD) discussed in [21] is proposed.

First of all, let us recall that a structuring element of size k, $B(k)$, can be considered as a k-fold dilation by the structuring element of unit size B [1], i.e.:

$$B(k)=\underbrace{B\oplus B\oplus\cdots\oplus B}_{k} \quad k=1,2,\ldots \qquad (6)$$

A more general decomposition of $B(k)$ is suggested in [10]:

$$B(k)=B_1\oplus B_2\oplus\cdots\oplus B_k \quad k\ge 1 \quad B_1\subset B_2\subset\cdots\subset B_k \qquad (7)$$

where $B_1,B_2,\ldots,B_k$ are all symmetric compact sets in $\mathcal{Z}^2$ containing the origin. USD and PUSD [21] rely on (6) and (7).

We propose the following class of distance functions:

$$\begin{aligned} d_a(x,y) &= \inf\{k: X_k\uparrow y\} \quad k\in\mathcal{N} \\ X_k &= (X_{k-1}\oplus B_k)\cup S_k^+ \quad k\ge 1 \quad ; X_0=\{x\} \end{aligned} \qquad (8)$$

where S_k^+ is a suitable set of points that will be described later on. Another definition of distance function that resembles (8) is given below:

$$\begin{aligned} d_a(x,y) &= \inf\{k: X_k\uparrow y\} \quad k\in\mathcal{N} \\ X_k &= (X_{k-1}\oplus B_k)-S_k^- \quad k\ge 1 \quad ; X_0=\{x\} \end{aligned} \qquad (9)$$

where $-$ denotes set subtraction. The distance functions defined by (8) and (9) are metrics in $\mathcal{Z}^2$, if the sets S_k^+, S_k^- are appropriately selected. Having defined the distance functions approximating Euclidean distance in $\mathcal{Z}^2$, we proceed to the design of the proper sets S_k^+, S_k^- that are included in (8), (9) and increase the accuracy of the proposed distance functions compared to USD and PUSD in approximating the Euclidean distance.

When one constructs the Voronoi diagram of N points in $\mathcal{Z}^2$, the fast computation of the distance between any two points is needed. At the same time, it is essential the distance evaluated by any metric to be as close as possible to Euclidean distance (4). The computation of (5) does not solve the problem, since its computational cost is high. Furthermore, a recursive calculation of the distance function is highly desirable, because it allows region growing. At each step, all equidistant points from the region center x can be appended to this region. A further reduction in the computational cost can be achieved by using the symmetries of the Euclidean distance function. The symmetry relations are listed below:

$$\begin{aligned} &d(\mathbf{0},|y|e^{\jmath\phi})=d(\mathbf{0},|y|e^{-\jmath\phi})=d(\mathbf{0},|y|e^{\jmath(\pi/2\pm\phi)})= \\ &d(\mathbf{0},|y|e^{\jmath(\pi\pm\phi)})=d(\mathbf{0},|y|e^{\jmath(3\pi/2\pm\phi)}) \end{aligned} \qquad (10)$$

where $y=(y_1,y_2)\in\mathcal{Z}^2$ and $|y|=\sqrt{y_1^2+y_2^2}$. In (10), it is assumed that $x=\mathbf{0}=(0,0)$ without any loss of generality. The symmetries described in (10) imply that the equidistant points to any point x according to (5) can be calculated form the set of points that satisfy the inequalities $0\le arg(x,y)\le\pi/4$ yielding a reduction of the computational complexity to 1/8 of the total effort needed.

Let us select the structuring element required in (8) to be RHOMBUS [1,5]. Then, the set S_k^+ is given by:

$$\begin{aligned} S_k^+ = \{&w\in\mathcal{Z}^2: \ w\in[(X_{k-1}\oplus 2B)-(X_{k-1}\oplus B)] \\ &\text{and } d_z(w,x)=k\} \end{aligned} \qquad (11)$$

where $2B=B\oplus B$. If S_k^+ is chosen as in (11), then X_k given by (8) implements a recursive method for growing a disk centered on x in $\mathcal{Z}^2$. At each step k, the disk X_{k-1} is dilated by B and the points of S_k^+ defined in (11) are appended. The dilation operator by the RHOMBUS structuring element expands X_{k-1} in the horizontal and vertical direction. The appended points S_k^+ improve the discrete approximation of the disk. The sets S_k^+, $k=1,2,\ldots$ can be precomputed and stored. The storage space requirements can be reduced to 1/8, if (10) is taken into account. It is seen that the points in S_k^+ are not included in the set $X_{k-1}\oplus B$ and satisfy the condition $d_z(w,x)=k$. The resulting disk growing is shown in Figure 1. It can be seen that the circularity is very good. The points belonging to S_k^+, $k=6,8,9,11,\ldots$, are indicated by small circles.

If the SQUARE structuring element [1,5] is used in (8) and (11) instead of RHOMBUS, it is found that some points in $X_{k-1}\oplus B$ do not satisfy $d_z(w,x)=k$. Therefore, the set of points violating $d_z(w,x)=k$ should be subtracted from $X_{k-1}\oplus B$. This set of points is denoted by S_k^- and is defined as follows:

$$\begin{aligned} S_k^- = \{&w\in\mathcal{Z}^2: \ w\in[(X_{k-1}\oplus B)-X_{k-1}] \\ &\text{and } d_z(w,x)\ne k\} \end{aligned} \qquad (12)$$

By using (9) and (12), another recursive scheme for growing a disk centered on x in $\mathcal{Z}^2$ results.

The above-described methods are not of equal performance in terms of storage requirements. When (8) and (11) are used, the number of appended points versus disk radius increases almost linearly with radius. This number is small for most practical cases. When (9) and (12) are employed, the number of subtracted points versus disk radius has been found that also increases linearly with radius. In the later case, the rate of increase is more than double compared to the rate of increase in the former case. In the following section, an implementation of Voronoi tessellation that relies on the proposed class of distance functions will be treated.

3. VORONOI TESSELLATION

A new method for implementing the Voronoi diagram of a given set of distinct points $X \subset W \subseteq Z^2$ is described. The proposed method finds the Voronoi regions of $X \subset W$ rather than the Voronoi edges and vertices. It uses iterative region growing of the m-Voronoi region $N_m(i)$ of each point $x_i \in X$. $N_m(i)$ is the set of points in W that have already been appended to x_i during the m previous growing steps. When two or more m-Voronoi regions collide, the collision points form subsets of the Voronoi polygons and the growing stops in that direction. This procedure is repeated until no further growth is possible in W.

Let $D_m(i)$ be the *m-neighborhood* of x_i, i.e., the set of points in W that are at a distance m from x_i. The m-neighborhood of x_i can be evaluated as follows:

$$D_m(i) = \{x \in W : x \in (X_m(i) - X_{m-1}(i))\} \qquad (13)$$

where

$$X_m(i) = \begin{cases} (\{x_i\} \oplus B) \cup S_1^+ & m = 1 \\ (X_{m-1}(i) \oplus B) \cup S_m^+ & m > 1 \end{cases} \qquad (14)$$

The points that belong to the m-neighborhood of a given point $x_i \in X$ and to the m-neighborhood of another point $x_j \in X$, $j \neq i$ are called *m-border points* of x_i. The set of m-border points of x_i, denoted by $e_m(i)$, is given by:

$$e_m(i) = \{x \in (W - X) : x \in \bigcup_{x_j \in (X - \{x_i\})} (D_m(i) \cap D_m(j))\} \; j \neq i \qquad (15)$$

Let $n_m(i)$ be the set of points in $(W - X)$ which belong to the m-neighborhood of a point $x_i \in X$ but at the same time do not belong to a k-neighborhood of another point $x_j \in X$ $j \neq i$ where $k \leq m$, i.e.:

$$n_m(i) = \{x \in (W - X) : x \in (n_{m-1}(i) \oplus B) \; ; \; x \notin e_m(i) \; , \; x \notin N_{m-1}(j) \; \forall x_j \in (X - \{x_i\})\} \quad m \geq 1 \qquad (16)$$

where $n_0(i) = \{x_i\}$. Then, the *m-Voronoi region* of x_i is obtained by:

$$N_m(i) = N_{m-1}(i) \cup n_m(i) \; ; \; N_0(i) = \emptyset \qquad (17)$$

If $k_{max}(i)$ denotes the step in which the point x_i cannot grow further, the Voronoi region $V(i)$ of the point x_i is simply:

$$V(i) = N_{k_{max}(i)}(i) \qquad (18)$$

Therefore, the union of the $k_{max}(i)$-Voronoi regions for all points in X yields the Voronoi diagram of the set X. The set containing the *boundary points* of a Voronoi region $V(i)$ is denoted by $F(i)$ and can be easily evaluating as follows:

$$F(i) = \{x \in (W - X) : x \in (N_{k_{max}(i)}(i) - N_{k_{max}(i)-1}(i))\} \qquad (19)$$

The Voronoi edges and vertices can be found implicitly. The Voronoi region of each point in $x_i \in X$ defines a Voronoi edge along its boundary line, i.e., $x \in F(i)$. The set of points $x \in e_m(i)$ found during the iterative process also belong to the Voronoi edges. In order to determine the Voronoi vertices, the set of points in X that are adjacent to $x_i \in X$ has to be found. We say that a point $x_j \in X$ is adjacent to x_i, if a distance value m (e.g., iteration of the growing procedure) exists such that the m-neighborhood of x_i, $D_m(i)$, hits the $m-1$-neighborhood of x_j, $D_{m-1}(j)$. Let $J_m(i)$ denote the set of adjacent points to x_i at the m iteration. It is given by:

$$J_m(i) = \{x_j \in (X - x_i) : D_m(i) \cap D_{m-1}(j) \neq \emptyset\}, \quad m \geq 1 \qquad (20)$$

The union of $J_m(i)$ for all $m \in [1, k_{max}(i)]$, yields the set adjacent points to x_i. Then, the Voronoi vertices are border points where three or more neighboorhoods of adjacent points are met or the points where three of more boundary lines are met.

By assuming that each growth step has the same computational cost, the computational complexity of the proposed implementation of Voronoi tessellation depends only on the number of growth steps executed. The number of growth steps, in the worst case, depends on the half of the greatest distance between the points in X, denoted by $d/2$. When the number N of the points in X increases, $d/2$ decreases if a uniform distribution of the points in X is assumed. This means that the proposed implementation, has a complexity inversely proportional to N. In general, the computational complexity of the method proposed depends on the distribution of the points in X. The more uniform the distribution the faster the method is.

4. SIMULATION EXAMPLES

The implementation of Voronoi tessellation described above has been employed in the derivation of Delauney triangulation. The Delauney triangulation is obtained by joining the object corners for each pair of adjacent Voronoi regions [3]. The Delauney triangulation of a polygonal object X is shown in Figure 2. Having defined how adjacent Voronoi regions are determined (e.g. by using (20)), we proceed to the computation of the set of object corners. The set X_c of the corners of this object can be obtained by morphological operations as follows. Let X_B denote *set opening* [5], i.e.,

$$X_B = \bigcup_{B_z} \{B_z : B_z \subset X\} = (X \ominus B^s) \oplus B \qquad (21)$$

Then, X_c is given by:

$$X_c = X_E \cup X_{E'} \quad X_E = X - X_B \quad X_{E'} = X^c - (X^c)_B \qquad (22)$$

where X^c is the complement of X with respect to W. The corner set X_c is used to obtain the Voronoi tessellation of X and subsequently its Delauney triangulation.

5. REFERENCES

[1] J. Serra, *Image Analysis and Mathematical Morphology*. New York: Academic Press, 1982.

[2] N. Ahuja and B.J. Schachter, *Pattern Models*. New York: J. Wiley, 1983.

[3] F.P. Preparata and M.I. Shamos, *Computational Geometry*. New York: Springer-Verlag, 1985.

[4] M.D. Levine, *Vision in Man and Machine*. New York: McGraw-Hill, 1985.

[5] I. Pitas and A.N. Venetsanopoulos, *Nonlinear Digital Filters: Principles and Applications*. Boston: Kluwer Academic, 1990.

[6] A. Rosenfeld and J. Pfaltz, "Distance Functions on Digital Pictures," *Pattern Recognition*, vol. 1, no. 1, pp. 33–61, 1968.

[7] C. Lantuejoul, "Skeletonization in Quantitative Metallography," in *Issues in Digital Image Processing* (R.M. Haralick and J.C. Simon, eds.), Maryland: Sijthoff and Noordhoff, 1980.

[8] P.E. Danielsson, "Euclidean Distance Mapping," *Computer Graphics and Image Processing*, vol. 14, pp. 227–248, 1980.

[9] G. Borgefors, "Distance Transformations in Arbitrary Dimensions," *Comput. Vision, Graphics, Image Proc.*, vol. 27, pp. 321–345, 1984.

[10] P. Maragos, "A unified theory of translation-invariant systems with applications to morphological analysis and coding of images," Ph.D. dissertation, School Electr. Enginr., Georgia Inst. Technology, Atlanta, GA, July 1985.

[11] G. Borgefors, "Distance Transformations in Digital Images," *Comput. Vision, Graphics, Image Proc.*, vol. 34, pp. 344–371, 1986.

[12] A.M. Vossepoel, "A Note on 'Distance Transformations in Digital Images'," *Comput. Vision, Graphics, Image Proc.*, vol. 43, pp. 88–97, 1988.

[13] P.P. Das and B.N. Chatterji, "A Note on 'Distance Transformations in Arbitrary Dimensions'," *Comput. Vision, Graphics, Image Proc.*, vol. 43, pp. 368–385, 1988.

[14] Q.Z. Ye, "The Signed Euclidean Distance Transform and Its Applications," in *Proc. 9th Int. Conf. on Pattern Recognition*, pp. 495–499, Rome, Italy 1988.

[15] C. Arcelli and G. Sanniti di Baja, "A One-Pass Two-Operation Process to Detect the Skeletal Pixels on the 4-Distance Transform," *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 11, no. 4, pp. 411–414, April 1989.

[16] B.J.H. Verwer, P.W. Verbeek and S.T. Dekker, "An Efficient Uniform Cost Algorithm Applied to Distance Transforms," *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 11, no. 4, pp. 425–429, April 1989.

[17] J.E. Mazille, "Mathematical Morphology and Convolutions," *Journal of Microscopy*, vol. 156, pp. 3–13, 1989.

[18] A.L.D. Beckers and A.W.N. Smeulders, "A Comment on 'A Note on Distance Transformations in Digital Images'," *Comput. Vision, Graphics, Image Proc.*, vol. 47, pp. 89–91, 1989.

[19] L. Vincent and P. Soille, "Watersheds in Digital Spaces: An Efficient Algorithm Based on Immersion Simulations," *IEEE Trans. on Pattern Analysis and Machine Intelligence*, vol. 13, no. 6, pp. 583–598, June 1991.

[20] F.Y.-C. Shin and O.R. Mitchell, "A Mathematical Morphology Approach to Euclidean Distance Transformation," *IEEE Trans. on Image Processing*, vol. 1, no. 2, pp. 197–204, April 1992.

[21] Z. Zhou and A. N. Venetsanopoulos, "Analysis and Implementation of Morphological Skeleton Transforms," *Circuits Systems Signal Process.*, vol. 11, no.1, pp. 253–280, 1992.

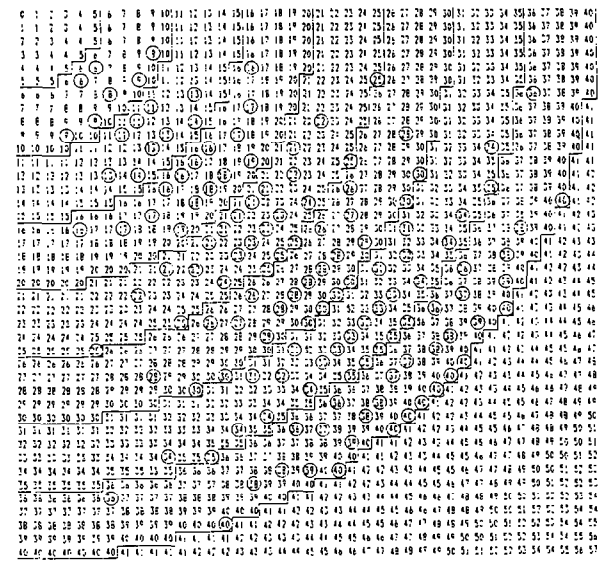

Figure 1: Euclidean disks in $\mathcal{Z}^2$

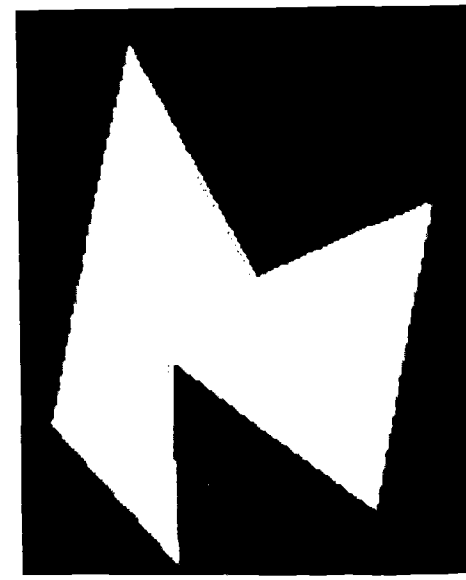

Figure 2: Delauney triangulation of a binary object.