Error analysis of recursive algorithms for computing the correlation matrix of the order statistics

C. Kotropoulos* I. Pitas
Department of Electrical Engineering, University of Thessaloniki
Thessaloniki 540 06, GREECE (fax: +30-31-274868)

1 Introduction

The computation of the expected values and moments of the order statistics has received much attention since the mid-1950's, because of the prominent position order statistics in the field of robust estimation [1]-[6],[9]. The computation of the second-order and product moments appearing in the correlation matrix of the order statistics is of fundamental importance, because the correlation matrix of the order statistics is involved in the design of L-filters both single-channel [7,8,10] as well as multichannel ones [13]. Furthermore, a need for computing the correlation matrix of the order statistics is also arisen when the assessment of the performance of adaptive nonlinear filters based on Least Mean Squares (LMS) or Recursive Least Squares algorithms is to be pursued [11,12]. For example, the stability as well as the rate of convergence of LMS adaptive L-filters depend on the extreme eigenvalues of the correlation matrix of the order statistics.

Two approaches for the computation of the moments of the order statistics can be found in the literature [6]. The first approach is based on continuous parent distributions whereas the second one is based on discrete parent distributions. However, order-recursive algorithms constitute the core of the procedures used in both cases [6,9]. Although, the following analysis is focused on order-recursive algorithms for continuous parent distributions, the same conclusions can also be applied to order-recursive algorithms for discrete parent distributions. More specifically, when the length of the L-filter increases, the order-recursive algorithms become attractive, because of their computation speed. However, their accuracy becomes questionable. The limited accuracy is attributed to a serious propagation of accumulating errors. This fundamental remark has been observed early in the statistical community [2,3,6]. To the authors' knowledge, upper bounds on the absolute accumulating error in the evaluation of the diagonal and off-diagonal elements of the correlation matrix of the order statistics have not been derived rigorously.

The present paper focuses on the error analysis of an order-recursive algorithm for computing the correlation matrix of the order statistics. Its major contribution is in deriving upper bounds on the absolute accumulating error made in the computation of the elements of the correlation matrix of the order statistics due to round-off and integration errors.

The outline of this paper is as follows. The computation of the correlation matrix of the order statistics for continuous parent distributions is briefly reviewed in Section 2. Upper bounds on the absolute accumulating errors made in order-recursive algorithms for computing the correlation matrix of the order statistics are derived in Section 3. Numerical results are included and conclusions are drawn in Section 4.

2 Computation of the correlation matrix of the order statistics

Let $\mathbf{R}_M$ be the correlation matrix of the order statistics of M independent identically distributed random variables which might be the samples of white additive noise corrupting a constant signal or the corrupted signal values themselves inside an L-filter window of length M. The (i,j) element of the correlation matrix is given by $r_M(i,j) = \mathrm{E}[x_{(i)}x_{(j)}]$, $i,j = 1,\ldots,M$. Let us denote by $\mathbf{R}_m$ the correlation matrix of the order statistics when the L-filter length is m, $m = 1,\ldots,M$. An order-recursive algorithm for the calculation of the elements of $\mathbf{R}_M$, where M is the final L-filter length can be derived by exploiting the following recurrence relations [3]-[6]:

$$(i-1)r_m(i,i) + (m-i+1)r_m(i-1,i-1) = mr_{m-1}(i-1,i-1) \quad i = 2,\ldots,m \tag{1}$$

$$(i-1)r_m(i,j) + (j-i)r_m(i-1,j) + (m-j+1)r_m(i-1,j-1) = mr_{m-1}(i-1,j-1) \quad 1 \le i < j \le m \tag{2}$$

It is easily recognized that

*Supported by a scholarship from the State Scholarship Foundation of Greece and the Bodosaki Foundation

(i) $r_1(1,1) = \mathrm{E}[x^2]$

(ii) At each recursion $m = 2, \ldots, M$ only one simple integral is required to be evaluated by numerical integration. This integral is given by [6]:

$$r_m(1,1) = m \int_{-\infty}^{+\infty} x^2 \, [1 - F(x)]^{m-1} f(x) \, dx \quad (3)$$

(iii) At each recursion $m = 2, \ldots, M$ only $(m-1)$ double integrals are required to be evaluated by numerical integration. These integrals are given by [6]:

$$r_m(1,j) = \frac{m!}{(j-2)! \, (m-j)!} \int_{-\infty}^{+\infty} \int_{x}^{+\infty} xy [F(y) - F(x)]^{j-2} [1 - F(y)]^{m-j} f(x) f(y) \, dx \, dy \quad j = 2, \ldots, m \quad (4)$$

Therefore, the total number of the necessary simple and double integrations are $(M-1)$ and $M(M-1)/2$ respectively. These numbers are equal to the ones required by the straightforward nonrecursive computation of the elements of the correlation matrix of the order statistics stated in [7]. However, by using the order-recursive algorithm outlined above, $(M-1)$ correlation matrices are calculated by the same effort. Thus, if the comparison is made with respect to the average number of simple and double integrations per evaluated correlation matrix, the order-recursive algorithm is preferable to the nonrecursive one. Having briefly described the principles of order-recursive algorithms for the computation of the correlation matrix of the order statistics, we proceed to the analysis of the errors that are inherent in this class of algorithms.

3 Error analysis of the recursive computation of the correlation matrix

Although, the recursive computation of the correlation matrix is computationally attractive, it results in a serious propagation of accumulating errors with the increase of the L-filter length M. In the following, upper bounds on the accumulating errors present in the evaluation of the diagonal and off-diagonal elements of $\mathbf{R}_M$ are derived, when floating point arithmetic is used. First, the elements which are obtained by simple numerical integration and affect a diagonal element are determined and then a bound on the accumulating error is derived. A similar approach is used for the off-diagonal elements.

It can be proven by mathematical induction that:

Proposition 1 *The (i,i) element of the correlation matrix $\mathbf{R}_M$ is given by:*

$$r_M(i,i) = \frac{1}{(i-1)!} \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} \frac{\pi_{i-1}(M)}{M-i+k+1} \, r_{M-i+k+1}(1,1) \quad (5)$$

where we assume that $r_{M-i+k+1}(1,1)$, $k = 0, \ldots, i-1$ represent simple integrations and $\pi_l(M)$ is the $(l+1)$ factorial polynomial of M.

A similar equation to (5) can be found in [6, p.48].

Let ϵ_1 be the maximum absolute relative error made in the simple integrations required in (5), i.e.,

$$\epsilon_1 = \max \left| \frac{\mathcal{E}_P\{r_{M-i+k+1}(1,1)\}}{r_{M-i+k+1}(1,1)} \right| \quad k = 0, \ldots, i-1, \; i = 2, \ldots, M \quad (6)$$

where $\mathcal{E}_P\{r_{M-i+k+1}(1,1)\}$ is the absolute error made in the numerical evaluation of a simple integral using a quadrature formula of P points. Then,

Proposition 2 *The absolute accumulating error made in the evaluation of the diagonal elements of the correlation matrix due to round-off and integration errors can be bounded as follows:*

$$|\hat{r}_M(i,i) - r_M(i,i)| < \frac{2^{i-1}}{(i-1)!} \left[4(i-1)2^{-b} + \epsilon_1 \right] \pi_{i-1}(M) \, \mathrm{E}[x^2] \quad (7)$$

where $\hat{r}_M(i,i)$ is the evaluated (i,i) element, $r_M(i,i)$ is the exact (i,i) element and b is the length of the mantissa.

It is worth noting that the integration errors are treated in a similar way to the round-off errors.

Proof:

Any diagonal element $r_M(i,i)$ can be written in the form:

$$r_M(i,i) = \sum_{k=0}^{i-1} a_k \quad (8)$$

where $a_k = c_k \, r_{M-i+k+1}(1,1)$. The coefficients c_k are given by:

$$c_k = \frac{1}{(i-1)!} \binom{i-1}{k} \frac{\pi_{i-1}(M)}{M-i+k+1} \quad (9)$$

We shall assume that the error made in the evaluation of the term a_k is due to:

(a) the round-off errors in the coefficients c_k, and

(b) the errors in $r_{M-i+k+1}(1,1)$.

Let us denote the relative error made in the numerical evaluation of a single integral by e_r. Then, the inexact result of the computation is given by:

$$\hat{r}_{M-i+k+1}(1,1) = r_{M-i+k+1}(1,1)(1+e_r) \quad (10)$$

If the inexact term a_k is denoted by $\hat{a}_k$, then:

$$\hat{a}_k = \text{fl}[c_k]\, \hat{r}_{M-i+k+1}(1,1) \quad (11)$$

where fl[.] allows for floating-point arithmetic errors (e.g. round-off errors).

First, the round-off error made in the coefficient c_k is estimated. We maintain that the maximum number of product-terms in the binomial coefficient $\binom{i-1}{k}$, $k=0,\ldots,i-1$ is $(i-1)$. Let $i=2\rho+1$, $\rho \in \mathcal{N}$ ($\mathcal{N}$ is the set of natural numbers). For the above-mentioned binomial coefficient we have:

$$\binom{i-1}{k} = \begin{cases} \frac{(i-1)\cdots(i-k)}{k!} & \text{if } k \leq (i-1)/2 \\ \frac{(k+1)\cdots(i-1)}{(i-1-k)!} & \text{if } k > (i-1)/2 \end{cases} \quad (12)$$

It is seen that if $k \leq (i-1)/2$, the number of product terms in the numerator is k and an additional number of k product terms are added from the denominator. In total, $2k$ terms have resulted. For $k=\rho$, we have the maximum number of product terms, i.e., $(i-1)$ terms. With same reasoning, $2(i-1-k)$ terms result if $k > (i-1)/2$. In the later case, it is easily recognized that the number of product terms is less than $(i-1)$. For i even, i.e., $i=2\rho$, $\rho \in \mathcal{N}$, we have two binomial coefficients with the maximum number of product terms, those for $k=\rho-1$ and $k=\rho$.

Furthermore, the term $\pi_{i-1}(M)/(M-i+k+1)$ is a product of $(i-1)$ terms. In addition, the term $1/(i-1)!$ is interpreted as a product of $(i-1)$ terms. Therefore, c_k is a product of $3(i-1)$ terms. We claim that:

$$\text{fl}[c_k] = c_k(1+e_{c_k}) \text{ with } |e| < (3i-4)/2^{-b} \quad (13)$$

where b is the length of the mantissa.

By replacing (10) and (13) into (11) we obtain:

$$\hat{a}_k = a_k\,(1+e_{c_k})\,(1+e_r) = a_k\,(1+e_{c_k} + + e_r + \mathcal{O}(e^2)) \simeq a_k\,(1+e_{c_k}+e_r) \quad (14)$$

If ϵ_1 is an upper bound for e_r, then:

$$|e_{a_k}| = |e_{c_k} + e_r| < (3i-4)\,2^{-b} + \epsilon_1 \quad (15)$$

We maintain that:

$$\text{fl}[\sum_{k=0}^{i-1} a_k] = \hat{a}_0(1+e_0) + \hat{a}_1(1+e_1) + \cdots + \hat{a}_{i-1}(1+e_{i-1}) \quad (16)$$

where $\hat{a}_k$ are the "noisy" sum-terms (14) and

$$|e_0| < (i-1)\,2^{-b} \quad |e_j| < (i-j)\,2^{-b} \quad j=1,\ldots,i-1 \quad (17)$$

The following simplifications can be made:

$$\begin{aligned}(1+e_{a_0})(1+e_0) &\simeq 1+e_{a_0}+e_0 \\ &= 1+e'_0 \\ (1+e_{a_j})(1+e_j) &\simeq 1+e_{a_j}+e_1 \\ &= 1+e'_j \quad j=1,\ldots,i-1 \end{aligned} \quad (18)$$

By using (15) and (18) we obtain:

$$\begin{aligned}|e'_0| &< |e_{a_0}| + |e_0| < [(i-1)+(3i-4)]\,2^{-b} + \epsilon_1 \\ &= (4i-5)\,2^{-b} + \epsilon_1 \\ |e'_j| &< |e_{a_j}| + |e_j| < [(i-j)+(3i-4)]\,2^{-b} + \epsilon_1 \\ &= (4i-j-4)\,2^{-b} + \epsilon_1 \end{aligned} \quad (19)$$

The accumulating error in (16) can be bounded as in numerical analysis literature [4]:

$$|\text{fl}[\sum_{k=0}^{i-1} a_k] - \sum_{k=0}^{i-1} a_k| < [4(i-1)\,2^{-b} + \epsilon_1] \sum_{k=0}^{i-1} |a_k| \quad (20)$$

An upper bound for the last term in (20) can be derived as follows:

$$\begin{aligned}\sum_{k=0}^{i-1} |a_k| &= \frac{\pi_{i-1}(M)}{(i-1)!} \sum_{k=0}^{i-1} \binom{i-1}{k} \\ &\frac{r_{M-i+k+1}(1,1)}{M-i+k+1} < \frac{\pi_{i-1}(M)}{(i-1)!} \sum_{k=0}^{i-1} \binom{i-1}{k} \\ &\text{E}[x^2] = \frac{2^{i-1}}{(i-1)!}\,\pi_{i-1}(M)\,\text{E}[x^2] \end{aligned} \quad (21)$$

where we have used that:

$$\begin{aligned} r_{M-i+k+1}(1,1) &< \sum_{l=1}^{M-i+k+1} r_{M-i+k+1}(l,l) \\ &= (M-i+k+1)\,\text{E}[x^2] \text{ and} \end{aligned} \quad (22)$$

$$\sum_{k=0}^{i-1} \binom{i-1}{k} = 2^{i-1} \quad (23)$$

By replacing (21) into (20) we obtain that:

$$|\hat{r}_M(i,i) - r_M(i,i)| < \frac{2^{i-1}}{(i-1)!}\,[4(i-1)\,2^{-b} + +\epsilon_1]\,\pi_{i-1}(M)\,\text{E}[x^2] \quad (24)$$

Q.E.D.

The elements of the correlation matrix of the order statistics obtained by a numerical computation of a double integral and affect the off-diagonal elements of $\mathbf{R}_M$ are determined subsequently.

Proposition 3 *The (i,j) elements of the correlation matrix $\mathbf{R}_M$ are given by:*

$$r_M(i,j) = \frac{1}{(i-1)!} \sum_{l=1}^{i} (-1)^{l-1} \binom{i-1}{l-1}$$
$$\pi_{i-l-1}(M) \sum_{n=1}^{l} \binom{l-1}{n-1} \prod_{s=1}^{l-n} (M-j+s)$$
$$\prod_{t=2}^{n} (j-i+t-2)\; r_{M-i+l}(1, j-i+n)$$
$$i = 2, \ldots, M-1, \;\; j > i \tag{25}$$

where $\pi_{-1}(M) = 1$.

Proposition 3 can be proved by mathematical induction. Due to lack of space, the proof of Proposition 3 will be omitted. An equation equivalent to (25) having a different arrangement of its terms also appears in [2].

Let ϵ_2 be the maximum absolute relative error made in the double integrations required in (25), i.e.,

$$\epsilon_2 = \max \left| \frac{\mathcal{E}_{PQ}\{r_{M-i+l}(1, j-i+n)\}}{r_{M-i+l}(1, j-i+n)} \right|$$
$$i = 2, \ldots, M-1, \;\; j = i+1, \ldots, M$$
$$l = 1, \ldots, M, \;\; n = 1, \ldots, l \tag{26}$$

where $\mathcal{E}_{PQ}\{r_{M-i+l}(1, j-i+n)\}$ denotes the absolute error made in the numerical evaluation of a double integral (4) using a quadrature formula of $P \times Q$ points. We maintain that :

Proposition 4 *The absolute error in the evaluation of the off-diagonal elements of $\mathbf{R}_M$ due to round-off and integration errors is bounded as follows:*

$$|\hat{r}_M(i,j) - r_M(i,j)| < \frac{i(i+1)}{2(i-1)!} \pi_{i-2}(M)$$
$$[4(i-1)2^{-b} + \epsilon_2] \sum_{l=1}^{i} \binom{i-1}{l-1}$$
$$\sum_{n=1}^{l} \binom{l-1}{n-1} |r_{M-i+l}(1, j-i+n)| \tag{27}$$

Proof:

If ϵ_2 is the maximum absolute relative error made in the evaluation of the required double integrals, then the relative error e_r in a numerical evaluation of a double integral is bounded by ϵ_2, i.e., $e_r \leq \epsilon_2$. It is recognized that any element $r_M(i,j)$, $j > i$ given by (25) is a sum of $i(i+1)/2$ product terms. Let us adopt the following notation for the product terms involved in the computation of $r_M(i,j)$:

$$a_{11} \; ; \; a_{21}, a_{22} \; ; \; \ldots \; ; \; a_{l1}, a_{l2}, \ldots, a_{ln}, \ldots, a_{ll} \; ;$$
$$a_{11}, \ldots, a_{ii} \tag{28}$$

The term a_{11} is given by:

$$a_{11} = \frac{1}{(i-1)!} \pi_{i-2}(M) r_{M-i+1}(1, j-i+1) \tag{29}$$

Let $e_{a_{11}}$ be the relative error made in its evaluation. Consequently, its actual (inexact) value is defined as:

$$\hat{a}_{11} = a_{11}(1 + e_{a_{11}}) \tag{30}$$

where:

$$|e_{a_{11}}| \leq [2(i-1) - 1]\, 2^{-b} + \epsilon_2$$
$$< 2(i-1)\, 2^{-b} + \epsilon_2 \tag{31}$$

For the term a_{21} given by:

$$a_{21} = -\frac{(i-1)}{(i-1)!} \pi_{i-3}(M)\,(M-j+1)$$
$$r_{M-i+2}(1, j-i+1) \tag{32}$$

we obtain:

$$|e_{a_{21}}| \leq [2(i-2) - 1]\, 2^{-b} + \epsilon_2 \tag{33}$$

In a similar manner, for the term a_{22} defined as follows:

$$a_{22} = -\frac{(i-1)}{(i-1)!} \pi_{i-3}(M)\,(j-i)\, r_{M-i+2}(1, j-i+2) \tag{34}$$

it can be shown that

$$|e_{a_{22}}| \leq 2(i-2)\, 2^{-b} + \epsilon_2 \tag{35}$$

In general, for the product term a_{ln}, $l = 3, \ldots, i$, $n = 1, \ldots, l$:

$$a_{ln} = \frac{(-1)^{l-1}}{(i-1)!} \binom{i-1}{l-1} \pi_{i-l-1}(M)$$
$$\binom{l-1}{n-1} \prod_{s=1}^{l-n} (M-j+s) \prod_{t=2}^{n} (j-i+t-2)$$
$$r_{M-i+l}(1, j-i+n) \tag{36}$$

we have:

$$\hat{a}_{ln} = a_{ln}(1 + e_{a_{ln}}) \;\; \text{with}$$
$$|e_{a_{ln}}| \leq [3(i-1) + (l-1)]\, 2^{-b} + \epsilon_2 \tag{37}$$

It can be seen that the maximum error occurs for $l = i$. As a result, we have:

$$|e_{a_{in}}| \leq 4(i-1)\, 2^{-b} + \epsilon_2 \tag{38}$$

Comparing (38) with (31), (33) (35), we claim that the relative error in each product term can be bounded as follows:

$$|e| \leq 4(i-1)\, 2^{-b} + \epsilon_2 \tag{39}$$

Therefore, the following inequality for the accumulating error in the evaluation of $r_M(i,j)$ can be written:

$$|\hat{r}_M(i,j) - r_M(i,j)| < \frac{i(i+1)}{2}[4(i-1)\,2^{-b} + \\ +\epsilon_2]\sum_{l=1}^{i}\sum_{n=1}^{l}|a_{ln}| \quad (40)$$

Since

$$\pi_{i-l+1}(M)\prod_{s=1}^{l-n}(M-j+s)\prod_{t=2}^{n}(j-i+t-2) \\ < \pi_{i-2}(M) \quad (41)$$

the inequality (40) can be rewritten as:

$$|\hat{r}_M(i,j) - r_M(i,j)| < \frac{i(i+1)}{2(i-1)!}\,\pi_{i-2}(M) \\ [4(i-1)\,2^{-b}+\epsilon_2]\sum_{l=1}^{i}\binom{i-1}{l-1} \\ \sum_{n=1}^{l}\binom{l-1}{n-1}|r_{M-i+l}(1,j-i+n)| \quad (42)$$

Q.E.D.

4 Numerical Results and Conclusions

The dominating factor in the bound (7) on the absolute accumulating error in the diagonal elements is the i-factorial polynomial of M, $\pi_{i-1}(M)$. Table 1 highlights its dominance. It is seen that this factor results in a serious amplification of the round-off and integration errors.

A similar consideration is made for the bound (27) on the accumulating error present in the off diagonal elements. The weighting coefficient on the maximum absolute relative error in the double integrations for $M = 5$ and a Gaussian distributed random variable x is tabulating in Table 2. The table [1, p. 417] has been used to calculate the entries of Table 2. In general, the numerical evaluation of a double integral results in relatively large errors. The recursive nature of the computation (2) amplifies these errors.

The noise sensitivity analyzed above may result in a violation of the positive semidefinite character of the evaluated correlation matrices for large filter lengths. In conclusion, arithmetic errors is the cost of the fast correlation matrix calculations by using recursive techniques.

Acknowledgments

The research reported in this paper has been supported by ESPRIT Basic Research Action project NAT (7130).

References

[1] D. Teichroew, "Tables of expected values of order statistics and products of order statistics for samples of size twenty and less from the normal distribution," *Ann. Math. Statist.*, vol. 27, pp. 364–387, 1956.

[2] K.S. Srikantan, "Recurrence relations between the pdf's of order statistics and some applications," *Ann. Math. Statist.*, vol. 33, pp. 169–177, 1962.

[3] Z. Govindarajulu, "On moments of order statistics and quasi-ranges from normal populations," *Ann. Math. Statist.*, vol. 34, pp. 633–651, 1963.

[4] E. Isaacson and H.B. Keller, *Analysis of Numerical Methods.* New York: J. Wiley, 1965.

[5] P.C. Joshi, "Recurrence relations for the mixed moments of order statistics," *Ann. Math. Statist.*, vol. 42, no.3, pp. 1096–1097, 1971.

[6] H.A. David, *Order Statistics.* New York: J. Wiley, 1980.

[7] A.C. Bovik, T.S. Huang and D.C. Munson, Jr., "A generalization of median filtering using linear combinations of order statistics," *IEEE Trans. Acoust. Speech and Signal Processing*, vol. ASSP 31, no. 6, pp. 1342–1350, December 1983.

[8] F. Palmieri and C.G. Boncelet, Jr., "Optimal MSE linear combination of order statistics for restoration of Markov processes," in *Proc. 20th Ann. Conf. Info. Sci. Syst.*, Princeton, NJ, March 1986.

[9] C.G. Boncelet, Jr., "Algorithms to compute order statistic distributions," *SIAM J. Sci. Stat. Comput.*, vol. 8, no. 5, pp. 868–876, September 1987.

[10] L. Naaman and A.C. Bovik, "Least-squares order statistic filters for signal restoration," *IEEE Trans. on Circuits and Systems*, vol. CAS-38, no. 3, pp. 244–257, March 1991.

[11] I. Pitas and A.N. Venetsanopoulos, "Adaptive filters based on order statistics," *IEEE Transactions on Signal Processing*, vol. 39, no. 2, pp. 518–522, February 1991.

[12] C. Kotropoulos and I. Pitas, "Constrained adaptive LMS L-filters," *Signal Processing*, vol. 26, no. 3, pp. 335–358, March 1992.

[13] C. Kotropoulos and I. Pitas, "Multichannel L-filters based on M-ordering," in Proc. of *IEEE Winter Workshop on Nonlinear Digital Signal Processing*, Tampere, Finland, 1993.

Table 1: Weighting coefficient on the maximum absolute relative error in simple integrations ϵ_1, for $M = 5$ and $E[x^2] = 1$

i	1	2	3	4	5
$\frac{2^{i-1}}{(i-1)!}$	1	2	2	4/3	2/3
$\pi_{i-1}(M)$	5	20	60	120	120
$\frac{2^{i-1}}{(i-1)!}\ \pi_{i-1}(M)$	5	40	120	160	80

Table 2: Weighting coefficient on the maximum absolute relative error in double integrations ϵ_2, for $M = 5$ and Gaussian pdf

(i,j)	(2,3)	(2,4)	(2,5)	(3,4)	(3,5)	(4,5)
$\frac{i(i+1)}{2(i-1)!}$	3			3		5/3
$\pi_{i-2}(M)$	5			20		60
$\sum_l \sum_m \{\}$	1.5	0.7658	2.7031	3.24	5.1228	11.819
total	22.5	11.487	40.546	194.4	307.36	1181.8