On the Theoretical Properties of L_p Mean Comparators

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ABSTRACT

Digital implementations of sorting networks that rely on a Digital Signal Processor core are not as efficient as their analog counterparts. This paper deals with the L_p comparators for which simple analog implementations exist. From a statistical point of view, the L_p comparators are based on the nonlinear means. Their probability density function and the first and second-order moments are derived for independent uniformly distributed inputs. L_p comparators provide estimates of the minimum and maximum of their inputs. Therefore, they introduce errors. A proper approach to compensate for the estimation errors is proposed.

1 INTRODUCTION

Sorting operations are estimated to account for over 25% of processing time for all computations [1]. Sorting is the basic operation employed in order statistics filters that constitute effective techniques for image/signal processing due to their statistical and robustness properties. Besides signal processing, other numerous applications of sorting can be found, e.g., in database management, communication networks, multiaccess memory, multiprocessors, shared disks, etc. Sorting algorithms have been extensively explored in the past few decades. Sorting networks are special cases of sorting algorithms. A sorting network has N inputs $x_1, \ldots, x_N$ and N outputs $x_{(1)}, \ldots, x_{(N)}$, where $x_{(i)}$ denotes the i-th order statistic of the set $\{x_1, \ldots, x_N\}$. That is, $x_{(1)}$ denotes the smallest element of the set, while $x_{(N)}$ denotes the largest element. Two of the most commonly used algorithms is the odd-even transposition network [6] and the Batcher's bitonic sorter [2]. The basic functional unit of a sorting network is the comparator, which receives two numbers at its inputs and presents their maximum and minimum at its outputs. Recently, a sorting network is shown to be a wave digital filter realization of an N-port memoryless nonlinear classical network [4].

The order statistics filters employ usually a Digital Signal Processor core. However, sorting is a computationally expensive operation, and a large area and power reduction can be obtained with simpler analog implementations [3]. This paper deals with the theoretical properties of the L_p comparator. Sorting networks based on L_p comparators were first proposed in [5]. However, L_p comparators are "noisy" comparators. Therefore, we have to compensate for the errors that are introduced by the L_p comparators, before we replace the conventional comparators in a sorting network with the proposed L_p comparators. To devise such an error compensation algorithm, first the statistical properties of the L_p comparators are explored and compared against those of the min-max comparators. Then, we propose a simple error compensation algorithm and we derive theoretically the gain that is obtained, when L_p comparators employing error compensation are used. Accordingly, the present paper extends the previously reported work [5].

2 L_p COMPARATORS

In this section, the L_p comparator is defined and its statistical properties are derived for independent uniformly distributed input samples. The L_p comparator employs nonlinear L_p and L_{-p} means with two inputs to estimate the minimum and maximum of two input samples, respectively [6], i.e.,

$$\hat{x}_{(1)} = L_{-p}(x_1, x_2) = \left(\frac{x_1^{-p} + x_2^{-p}}{2} \right)^{-1/p} \quad (1)$$

$$\hat{x}_{(2)} = L_p(x_1, x_2) = \left(\frac{x_1^{p} + x_2^{p}}{2} \right)^{1/p} \quad (2)$$

where p is a positive rational number different than 1, i.e., $p \in \mathbb{Q}^+ - \{0, 1\}$. In contrast to classical max/min comparators, whose output is one of input samples, an L_p comparator provides estimates of the minimum and the maximum sample. If x_i, $i = 1, 2$, are independent random variables (RVs) uniformly distributed in the interval $[0, L]$, the probability density function (pdf) of the random variable $z = L_p(x_1, x_2)$ is given by:

$$f(z) = \begin{cases} \frac{2^{2/p}}{L^2\,p}\,B(1/p,1/p)\,z & \text{if} \quad 0 \le z < 2^{-1/p}\,L \\ \frac{2^{2/p}}{L^2\,p}\,B(1/p,1/p)\,z\,\left(2\,I_{\frac{L^p}{2z^p}}(1/p,1/p) - 1\right) & \\ \quad \text{if} \quad 2^{-1/p}\,L \le z < L & \\ 0 \quad \text{otherwise} & \end{cases} \tag{3}$$

where $B(\cdot)$ denotes the Beta function and $I_x(a,b)$ is the incomplete Beta function defined as [8]:

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}\,(1-t)^{b-1}\,dt. \tag{4}$$

The derivation of (3) is based on the determination of the pdf of functions of one and two RVs [7]. For $p = 2$, we obtain:

$$f(z) = \begin{cases} \frac{\pi}{L^2}\,z & \text{if} \quad 0 \le z < \frac{1}{\sqrt{2}}\,L \\ \frac{2}{L^2}\arcsin\left(\frac{L^2 - z^2}{z^2}\right) & \text{if} \quad \frac{1}{\sqrt{2}}\,L \le z < L \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

The pdf of RV z is plotted for $p = 2, 5, 8$ in Figure 1. For

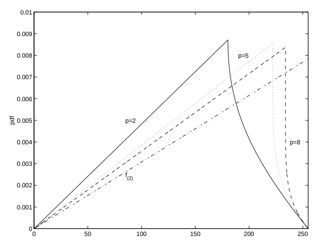


Figure 1: Probability density function of the RV $z = L_p(x_1, x_2)$ for $p = 3, 5, 8$, when x_1 and x_2 are independent RVs uniformly distributed in the interval $[0, L]$.

completeness, the pdf of the RV $x_{(2)}$ for uniform parent distribution in the interval $[0, L]$ and $N = 2$ is included:

$$f_{(2)}(x) = \frac{2}{L}\,\frac{x}{L}, \quad 0 \le x \le L. \tag{6}$$

It can be shown that the expected value and the mean square value of the RV z is given by:

$$\mathrm{E}\{z\} = \frac{2\,L}{3}\left(1 - \frac{2^{-1/p}}{p}\int_0^1 t^{1/p}\,(1+t)^{1/p-1}\,dt\right) \tag{7}$$

$$\mathrm{E}\{z^2\} = \frac{L^2}{2}\left(1 - \frac{2^{-2/p}}{p}\int_0^1 t^{1/p}\,(1+t)^{2/p-1}\,dt\right), \tag{8}$$

respectively. For $p = 2$, (7) and (8) yield:

$$\mathrm{E}\{z\} = 2^{-1/p}\,\frac{L}{3}\left(\sqrt{2} - \ln\tan\left(\frac{\pi}{8}\right)\right) \tag{9}$$

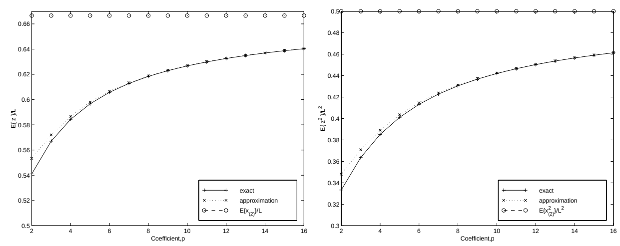
$$\mathrm{E}\{z^2\} = 2^{-2/p}\,\frac{2}{3}\,L^2. \tag{10}$$

The following approximate expressions for the first and second moment of RV z hold:

$$\mathrm{E}\{z\} \approx \frac{L}{2}\left[(1 + 2^{-1/p}) - \frac{B(1/p,1/p)}{6\,p}(3 - 2^{-1/p})\right] \tag{11}$$

$$\mathrm{E}\{z^2\} \approx 2^{-2/p}L^2\left[2^{1/p} + \frac{B(1/p,1/p)}{2p}\left(\frac{1}{2} - 2^{1/p}\right)\right]. \tag{12}$$

The expected value and the mean square value of the RV z for several values of the coefficient p are plotted in Figure 2(a) and (b), respectively. The approximate



(a) (b)

Figure 2: First and second moment of the RV $z = L_p(x_1, x_2)$ for several values of the coefficient p. (a) Expected value; (b) Mean square value.

values obtained by using (11) and (12) are overlaid for comparison purposes. It is seen that for $p > 8$ the values obtained by the approximate expressions are practically the same to those obtained by numerical integration of (7) and (8). The expressions in (7) and (8) should be compared to those of the order statistics for $N = 2$ and uniform parent distribution that are given by [9]:

$$\mathrm{E}\{x_{(2)}\} = \frac{2\,L}{3}, \qquad \mathrm{E}\{x_{(2)}^2\} = \frac{L^2}{2}. \tag{13}$$

It is obvious that the first and second moments of the RV z tend to those of the RV $x_{(2)}$ for large p.

Similarly, if x_i, $i = 1, 2$, are independent RVs uniformly distributed in the interval $[\epsilon, L]$, the pdf of the random variable $w = L_{-p}(x_1, x_2)$ is given by:

$$f_w(w) = \begin{cases} \frac{2^{-2/p}}{p\,(L-\epsilon)^2}\,w \int_{1-\frac{w^p}{2\,\epsilon^p}}^{\frac{w^p}{2\,\epsilon^p}} t^{-(1+\frac{1}{p})}(1-t)^{-(1+\frac{1}{p})}\,dt \\ \quad \text{if} \quad \epsilon < w \le 2^{1/p}\frac{\epsilon\,L}{(\epsilon^p + L^p)^{1/p}} \\ \frac{2^{-2/p}}{p\,(L-\epsilon)^2}\,w \int_{\frac{w^p}{2\,L^p}}^{1-\frac{w^p}{2\,L^p}} t^{-(1+\frac{1}{p})}(1-t)^{-(1+\frac{1}{p})}\,dt \\ \quad \text{if} \quad 2^{1/p}\frac{\epsilon\,L}{(\epsilon^p + L^p)^{1/p}} < w \le L \\ 0 \quad \text{otherwise.} \end{cases} \tag{14}$$

For $p = 2$, (14) is simplified to:

$$f_w(w) = \begin{cases} \dfrac{2}{(L-\epsilon)^2} \dfrac{w^2 - \epsilon^2}{\sqrt{2\epsilon^2 - w^2}} \\ \qquad \text{if} \quad \epsilon < w \le \sqrt{2}\dfrac{\epsilon\, L}{\sqrt{\epsilon^2 + L^2}} \\ \dfrac{2}{(L-\epsilon)^2} \dfrac{L^2 - w^2}{\sqrt{2L^2 - w^2}} \\ \qquad \text{if} \quad \sqrt{2}\dfrac{\epsilon\, L}{\sqrt{\epsilon^2 + L^2}} < w \le L \\ 0 \qquad \text{otherwise.} \end{cases} \tag{15}$$

The pdf of the RV $x_{(1)}$ for uniform parent distribution in the interval $[0, L]$ for $N = 2$ is given by:

$$f_{(1)}(x) = \frac{2}{L}\left(1 - \frac{x}{L}\right), \quad 0 \le x \le L. \tag{16}$$

For x_i, $i = 1, 2$, independent RVs uniformly distributed in the interval $[0.1, 1]$, the pdf of the L_{-p} comparator output is found by employing numerical integration and is plotted in Figure 3 for $p = 2, 4, 6$. The limit of the

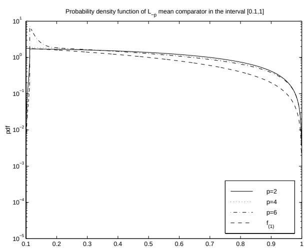


Figure 3: Probability density function of the RV $w = L_{-p}(x_1, x_2)$ obtained by numerical integration for $p = 2, 4, 6$, when x_1 and x_2 are independent RVs uniformly distributed in the interval $[0.1, 1]$.

expected value and the mean square value of the RV w for $\epsilon \to 0$ is given by:

$$\lim_{\epsilon\to 0} \mathrm{E}\{w\} = \frac{4}{3}\, 2^{1/p-1}\, L \int_0^{2^{-1/p}} \frac{t\, dt}{(1-t^p)^{1+1/p}} \tag{17}$$

$$\lim_{\epsilon\to 0} \mathrm{E}\{w^2\} = 2^{2/p-1}\, L^2 \int_0^{2^{-1/p}} \frac{t^2\, dt}{(1-t^p)^{1+1/p}}, \tag{18}$$

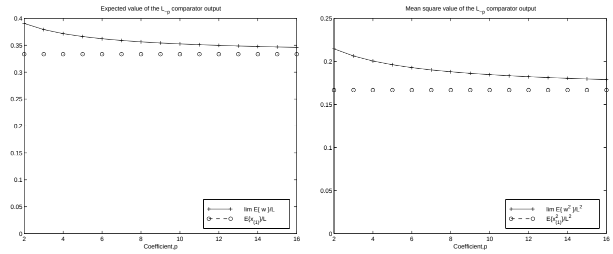
respectively. For $p = 2$, we obtain:

$$\lim_{\epsilon\to 0} \mathrm{E}\{w\} = \frac{4}{3}\, L\, (1 - \frac{1}{\sqrt{2}}) \tag{19}$$

$$\lim_{\epsilon\to 0} \mathrm{E}\{w^2\} = L^2\, (1 - \frac{\pi}{4}). \tag{20}$$

The expected value and the mean square value of the RV w for several values of the coefficient p are plotted in Figure 4(a) and (b), respectively. It can easily be verified that for p large, the first and the second moment of the RV w approximate those of the RV $x_{(1)}$, i.e.:

$$\mathrm{E}\{x_{(1)}\} = \frac{L}{3}, \qquad \mathrm{E}\{x_{(1)}^2\} = \frac{L^2}{6}. \tag{21}$$



(a) (b)

Figure 4: Limit of first and second moment of the RV $w = L_{-p}(x_1, x_2)$ for several values of the coefficient p when $\epsilon \to 0$. (a) Expected value; (b) Mean square value.

3 ERROR COMPENSATION

L_p comparators introduce errors. Let $e_{\max}(x_1, x_2) = x_{(2)} - \hat{x}_{(2)}$ denote the error introduced by the L_p comparator in the estimation of the maximum of two input samples. Then:

$$0 \le e_{\max}(x_1, x_2) \le \frac{1}{2}\, |x_2 - x_1|. \tag{22}$$

Similarly, let $e_{\min}(x_1, x_2) = x_{(1)} - \hat{x}_{(1)}$ denote the corresponding error in the estimation of the minimum of two input samples. It can easily be shown that:

$$-\frac{1}{2}\, |x_2 - x_1| \le e_{\min}(x_1, x_2) \le 0. \tag{23}$$

For x_i, $i = 1, 2$, independent RVs uniformly distributed in the interval $[0, L]$ it can be shown that the mean squared error (MSE) introduced by the L_p comparator is given by:

$$\mathrm{E}\{e_{\max}^2(x_1, x_2)\} = \frac{L^2}{2}\, \left\{1 + 2^{-2/p} \int_0^1 (1+t^p)^{2/p}\, dt - 2^{1-1/p} \int_0^1 (1+t^p)^{1/p}\, dt\right\}. \tag{24}$$

For $p = 2$, we obtain:

$$\mathrm{E}\{e_{\max}^2(x_1, x_2)\} = \frac{L^2}{2}\left(\frac{2}{3} - \frac{\ln(1+\sqrt{2})}{\sqrt{2}}\right). \tag{25}$$

The MSE of the L_p comparator is plotted for several values of the coefficient p in Figure 5. It is seen that the larger the coefficient p is, the smaller the MSE introduced by the L_p comparator becomes. Accordingly, for large values of the coefficient p, the L_p comparator converges to the max operator, as expected.

If x_i, $i = 1, 2$, are independent RVs uniformly distributed in the interval $[\epsilon, L]$, it can be shown that for $\epsilon \to 0$, the limit of the MSE of the L_{-p} comparator is:

$$\lim_{\epsilon\to 0} \mathrm{E}\{e_{\min}^2(x_1, x_2)\} = L^2\, \left\{\frac{1}{6} + 2^{2/p-1} \int_0^1 \frac{t^2}{(1+t^p)^{2/p}}\, dt - 2^{1/p} \int_0^1 \frac{t^2}{(1+t^p)^{1/p}}\, dt\right\}. \tag{26}$$

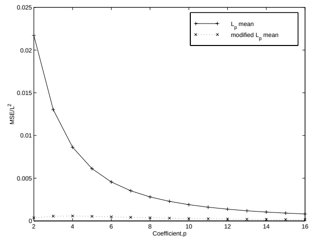


Figure 5: MSE of the L_p comparator and the modified L_p comparator in estimating the maximum of two independent input samples that are uniformly distributed in the interval $[0, L]$.

For $p = 2$, (26) is simplified to:

$$\lim_{\epsilon \to 0} \mathrm{E}\{e^2_{\min}(x_1, x_2)\} = L^2 \left(\frac{1}{6} - \frac{\pi}{4} + \frac{\sqrt{2}}{2} \ln(1 + \sqrt{2}) \right). \tag{27}$$

The MSE of the L_{-p} comparator is plotted for several values of the coefficient p in Figure 6 as well. It is seen that the larger the coefficient p is, the smaller the MSE introduced by the L_{-p} comparator becomes. Accordingly, for large values of the coefficient p, the L_{-p} comparator converges to the min operator, as expected.

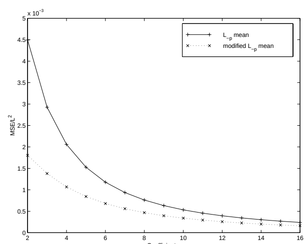


Figure 6: Limit of the MSE of the L_{-p} comparator and the modified L_{-p} comparator in estimating the minimum of two independent input samples that are uniformly distributed in the interval $[\epsilon, L]$, for $\epsilon \to 0$.

Next, we compensate for the MSE introduced by the L_p comparators for small p. We argue that the estimation error increases almost linearly with the absolute value of the difference between x_1, x_2 (i.e., their distance). Accordingly, we propose to modify the L_p comparator outputs as follows:

$$\tilde{x}_{(1)} = L_{-p}(x_1, x_2) - d\,|s|, \quad d > 0 \tag{28}$$

$$\tilde{x}_{(2)} = L_p(x_1, x_2) + c\,|s|, \quad c > 0 \tag{29}$$

where $s = x_2 - x_1$ and c, d are constants. The constants c and d can be chosen so that the $\mathrm{E}\{\tilde{e}^2_{\max}\}$ and $\lim_{\epsilon \to 0} \mathrm{E}\{\tilde{e}^2_{\min}\}$ is minimized, respectively. It can be shown that the optimal constants c, d are given by:

$$c = \frac{3}{2} + 3 \cdot 2^{-1/p} \int_0^1 (s-1)\,(1+s^p)^{1/p}\, ds \tag{30}$$

$$d = -\frac{1}{2} - 3 \cdot 2^{1/p} \int_0^1 \frac{s\,(s-1)}{(1+s^p)^{1/p}}\, ds, \tag{31}$$

respectively. The optimal constants c, d are plotted for several values of the coefficient p in Figure 7. The MSE

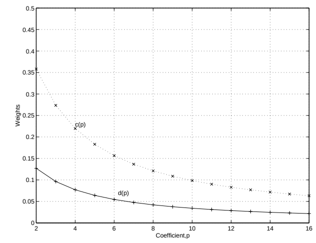


Figure 7: Optimal constants c, d that minimize the MSE between the modified L_p comparator output and the true maximum and minimum of two independent uniformly distributed samples, for several values of the coefficient p.

between the modified L_p comparator output (29) and the true maximum sample is given by:

$$\mathrm{E}\{\tilde{e}^2_{\max}\} = \mathrm{E}\{e^2_{\max}\} - \frac{c^2\, L^2}{6} \tag{32}$$

It is overlaid in Figure 5 for comparison purposes. Similarly, the MSE between the modified L_{-p} comparator output (28) and the true minimum sample, $\mathrm{E}\{\tilde{e}^2_{\min}\} = \mathrm{E}\{e^2_{\min}\} - \frac{d^2\, L^2}{6}$, is shown overlaid in Figure 6.

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